POLISH GROUPOIDS AND FUNCTORIAL COMPLEXITY

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Abstract. We introduce and study the notion of functorial Borel complexity for Polish groupoids. Such a notion aims at measuring the complexity of classifying the objects of a category in a constructive and functorial way. In the particular case of principal groupoids such a notion coincide with the usual Borel complexity of equivalence relations. Our main result is that on one hand for Polish groupoids with essentially treeable orbit equivalence relation, functorial Borel complexity coincides with the Borel complexity of the associated orbit equivalence relation. On the other hand for every countable equivalence relation E that is not treeable there are Polish groupoids with different functorial Borel complexity both having E as orbit equivalence relation. In order to obtain such a conclusion we generalize some fundamental results about the descriptive set theory of Polish group actions to actions of Polish groupoids, answering a question of Arlan Ramsay. These include the Becker-Kechris results on Polishability of Borel G-spaces, existence of universal Borel G-spaces, and characterization of Borel G-spaces with Borel orbit equivalence relations.

Contents

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1. Introduction

Classification of mathematical structures is one of the main components of modern mathematics. It is safe to say that most results in mathematics can be described as providing an explicit classification of a class of mathematical objects by a certain type of invariants.

In the last 25 years the notion of constructive classification has been given a rigorous formulation in the framework of invariant complexity theory. In this context a classification problem is regarded as an equivalence relation on a standard Borel space (virtually all classification problems in mathematics fit into this category). The concept of constructive classification is formalized by the notion of *Borel reduction*. A Borel reduction from an equivalence relation E on X to an equivalence relation E' on X' is a *Borel* function $f : X \to X'$ with the property that, for every $x, y \in X$,

$$xEy \text{ if and only if } f(x)E'f(y).$$

In other words f is a Borel assignment of complete invariants for E that are equivalence classes of E'. The existence of such a function can be interpreted as saying that classifying the objects of X' up to E' is at least as complicated as classifying the objects of X up to E. This offers a notion of comparison between the complexity of different classification problems.

Several natural equivalence relation can then be used as benchmarks to measure the complexity of classification problems. Perhaps the most obvious such a benchmark is the relation of equality $=_{\mathbb{R}}$ of real numbers. This gives origin to the basic dichotomy smooth vs. non-smooth: an equivalence relation is *smooth* if it is Borel reducible to $=_{\mathbb{R}}$. (The real numbers can here be replaced by any other uncountable standard Borel space.) Beyond smoothness the next fundamental benchmark is classifiability by countable structures. Here the test is Borel reducibility to the relation of isomorphism within some class of countable first order structures, such as (ordered) groups, rings, etc. Equivalently one can consider orbit equivalence relation associated with Borel actions of the Polish group S_∞ of permutations of $\mathbb{N}$. Replacing S_∞ with an arbitrary Polish group yields the notion of equivalence relation classifiable by orbits of a Polish group action.

This framework allows one to build a hierarchy between different classification problems. Many efforts have been dedicated to the attempt to draw a picture as complete as possible of classification problems in mathematics and their relative complexity. To this purpose powerful tools such as Hjorth's theory of turbulence [21] have been developed in order to disprove the existence of Borel reduction between given equivalence relations, and to distinguish between the complexity of different classification problems. This can be interpreted as a way to formally exclude the possibility of a full classification of a certain class of objects by means of a given type of invariants. For example the relation of isomorphism of simple separable C*-algebras has been shown to transcend countable structures in [14]; see also [45]. Similar results have been obtained for several other equivalence relations, such as affine homeomorphism of Choquet simplexes [14], conjugacy of unitary operators on the infinite dimensional separable Hilbert space [29], conjugacy of ergodic measure-preserving transformations of the Lebesgue space [16], conjugacy of homeomorphisms of the unit square [21], conjugacy of irreducible representations of non type I groups [20] or C*-algebras [11, 30], conjugacy and unitary equivalence of automorphisms of classifiable simple separable C*-algebras [31, 34], isometry of separable Banach spaces [37] and complete order isomorphism of separable operator systems. Furthermore the relations of isomorphism and Lipschitz isomorphisms of separable Banach spaces, topological isomorphism of (abelian) Polish groups, uniform homeomorphism of complete separable metric spaces [15], and the relation of completely bounded isomorphism of separable operator spaces [2] have been shown to be not classifiable by the orbits of a Polish group action (and in fact to have maximal complexity among analytic equivalence relations). An exhaustive introduction to invariant complexity theory can be found in [18].

Considering how helpful the theory of Borel complexity has been so far in giving us a clear understanding of the relative complexity of classification problems in mathematics, it seems natural to look at refinements to the notion of Borel reducibility, that can in some situations better capture the notion of explicit classification from the practice of mathematics. Such a line of research has been suggested in [12], where the results of the present paper has been announced. This is the case for example when the classification problem under consideration concerns a category. In this case it is natural to ask to the classifying map to be *functorial*, and to assign invariants not only to the objects of the category, but also to the *morphisms*. This is precisely what happens in many explicit examples of classification results in mathematics. In fact in many such examples the consideration of invariants of morphisms is essential to the proof. This is particularly the case in the Elliott classification program of simple C*-algebras, starting from Elliott's seminal paper of AF algebras [7]. Motivated by similar considerations, Elliott has suggested in [8] an abstract approach to classification by functors. In this paper we bring Elliott's theory of functorial classification within the framework of Borel complexity theory. For simplicity we consider only categories where every arrow is invertible, called groupoids. Such categories will be assumed to have a global Borel structure that is at least analytic, and makes the set of objects (identified with their identity arrows) a standard Borel space. In the particular case when between any two objects there is at most one arrow (principal groupoids) these are precisely the analytic equivalence relations. One can then consider the natural constructibility requirement for classifying functors, which is being Borel with respect to the given Borel structures. This gives rise to the notion of functorial Borel complexity, which in the particular case of principal groupoids is the usual notion of Borel complexity.

In this article we study such a notion of functorial Borel complexity for groupoids, focusing on the case of Polish groupoids. These are the groupoids where the Borel structure is induced by a topology that makes composition and inversion of arrows continuous and open, and has a basis of open sets which are Polish in the relative topology. These include all Polish groups, groupoids associated with Polish group actions, and locally compact groupoids [40, Definition 2.2.2]. The latter ones include the holonomy groupoids of foliations and the tangent groupoids of manifolds [40, Chapter 2], the groupoids of row-finite directed graphs [33], the localization groupoids of actions of countable inverse semigroups [40, Chapter 4]. The main results of the present paper assert that, for Polish groupoids with essentially countable equivalence relation, the existence of a Borel reducibility between the groupoids is equivalent to the Borel reducibility of the corresponding orbit equivalence relations. On the other hand for every countable equivalence relation E that is not treeable there are two Polish groupoid with orbit equivalence relation E that have distinct functorial Borel complexity; see Section 8. This shows that Borel reducibility of groupoids provides a finer notion of complexity than the usual Borel reducibility of equivalence relations. Having a finer notion of complexity is valuable, because it allows one to further distinguish between the complexity of problems that, in the usual framework, turn out to have the same complexity. An example of this phenomenon occurs in the classification problem for C*-algebras, where it turns out [9, 14, 45] that classifying arbitrary separable C*-algebras is as difficult as classifying the restricted class of C*-algebras that are considered to be well behaved (precisely the amenable simple C*-algebras, or even more restrictively the simple C*-algebras that can be obtained as direct limits of interval algebras).

In order to prove the above mentioned characterization of essentially treeable equivalence relations we will generalize some fundamental results of the theory of actions of Polish groups to actions of Polish groupoids, answering a question of Ramsay from [43]. These include the Becker-Kechris results on Polishability of Borel G-spaces [3, Chapter 5], existence of universal Borel G-spaces [3, Section 2.6], and characterization of Borel G-spaces with Borel orbit equivalence relation [3, Chapter 7]. The fundamental technique employed is a generalization of the *Vaught transform* [48] from actions of Polish groups to actions of Polish groupoids.

This paper is organized as follows: In Section 2 we recall some background notions, introduce the notation to be used in the rest, and state the basic properties of the Vaught transform for actions of Polish groupoids. In Section 3 we generalize the local version of Effros' theorem from Polish group actions to actions of Polish groupoids, and infer the Glimm-Effros dichotomy for Polish groupoids and Borel reducibility, refining results from [42]. Section 4 contains the proof of the Polishability result for Borel G-spaces, showing that any Borel G-space is isomorphic to a Polish G-space, where G is a Polish groupoid. A characterization for Borel G-spaces with Borel orbit equivalence relation is obtained as a consequence in Section 5. Section 6 contains the construction of a universal Borel G-space for a given Polish groupoid G, generalizing [3, Section 2.6]. Section 7 considers countable Borel groupoids, i.e. analytic groupoids with only countably many arrows with a given source. It is shown that every such a groupoid has a Polish groupoid structure compatible with its Borel structure. In particular all results about Polish groupoids apply to countable Borel groupoids. Finally in Section 8 the above mentioned characterization of essentially treeable equivalence relations in terms of Borel reducibility is proved.

This paper includes an appendix written by Anush Tserunyan. In such an appendix it is proved that the Effros Borel structure on the space of closed subsets of a Polish groupoid is standard. We are grateful to Anush Tserunyan for letting us include her result here.

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2. Notation and preliminaries

2.1. **Descriptive set theory.** A *Polish space* is a separable and completely metrizable topological space. Equivalently a topological space is Polish if it is T_1, regular, second countable, and *strong Choquet* [28, Theorem 8.18]. A subspace of a Polish space is Polish with respect to the subspace topology if and only if it is a G_δ [28, Theorem 3.11].

A *standard Borel space* is a space endowed with a σ-algebra which is the σ-algebra of Borel sets with respect to some Polish topology. An *analytic space* is a space endowed with a countably generated σ-algebra which is the image of a standard Borel space under a Borel function. A subset of a standard Borel space is analytic if it is an analytic space with the relative standard Borel structure. A subset of a standard Borel space is *co-analytic* if its complement is analytic. It is well known that for a subset of a standard Borel space it is equivalent being Borel and being both analytic and co-analytic [28, Theorem 14.7]. If X, Y are standard Borel space and A is a subset of $X \times Y$, then for $x \in X$ the section

$$\{y \in Y : (x, y) \in A\}$$

is denoted by A_x. The *projection* of A onto the first coordinate is

$$\{x \in X : A_x \neq \varnothing\},$$

while the *co-projection* of A is

$$\{x \in X : A_x = Y\}.$$

The projection of an analytic set is analytic, while the co-projection of a co-analytic set is co-analytic.

If X is a Polish space, then the space of closed subsets of X is denoted by $F(X)$. The *Effros Borel structure* on $F(X)$ is the σ-algebra generated by the sets

$$\{F \in F(X) : F \cap U \neq \varnothing\}$$

for $U \subset X$ open. This makes $F(X)$ a standard Borel space [28, Section 12.C].

Recall that a subset A of a Polish space X has the *Baire property* if there is an open subset U of X such that the symmetric difference $A \triangle U$ is meager [28, Definition 8.21]. It follows from [28, Corollary 29.14] that any analytic subset of X has the Baire property.

A topological space X is a *Baire space* if every nonempty open subset of X is not meager. Every completely metrizable topological space is a Baire space; see [28, Theorem 8.4].

If X, Y are standard Borel spaces, then we say that Y is *fibred* over X if there is a Borel surjection $p : Y \to X$. If $x \in X$, then the inverse image of x under p is called the x-fiber of Y and denoted by Y_x. If Y_0, Y_1 are fibred over X, then the fibred product

$$Y_0 * Y_1 = \{(y_0, y_1) : p_0(y_0) = p_1(y_1)\}$$

is naturally fibred over X. Similarly if $(Y_n)_{n\in\omega}$ is a sequence of Borel spaces fibred over X we define

$$\mathop{\Huge *}_{n\in\omega} Y_n = \{(y_n)_{n\in\omega} : p(y_n) = p(y_m) \text{ for } n, m \in \omega\}$$

which is again fibred over X. A *Borel fibred map* from Y_0 to Y_1 is a Borel function $\varphi : Y_0 \to Y_1$ which sends fibers to fibers, i.e. $p_1 \circ \varphi = p_0$.

If E is an equivalence relation on a standard Borel space X, then a subset T of X is a *transversal* for E if it intersects any class of E in exactly one point. A *selector* for E is a Borel function $\sigma : X \to X$ such that $\sigma(x)Ex$ for every $x \in X$ and $\sigma(x) = \sigma(y)$ whenever xEy.

2.2. **Locally Polish spaces.**

Definition 2.2.1. A *locally Polish space* is a topological space with a countable basis of open sets which are Polish spaces in the relative topology.

By [28, Theorem 8.18] a locally Polish space is T_1, second countable, and strong Choquet. Moreover it is a Polish space if and only if it is regular. It follows from [28, Lemma 3.11] that a G_δ subspace of a locally Polish space is locally Polish.

Suppose that X is a locally Polish space. Denote by $F(X)$ the space of closed subsets of X. The Effros Borel structure on $F(X)$ is the σ-algebra generated by the sets of the form

$$\{F : F \cap U \neq \varnothing\}$$

for $U \subset X$ open. It is shown in the Appendix that the Effros Borel structure on $F(X)$ is standard.

One can deduce from this that the Borel σ-algebra of X is standard. In fact the function

$$\begin{array}{rcl} X & \to & F(X) \\ x & \mapsto & \{x\} \end{array}$$

is clearly a Borel isomorphism onto the set $F_1(X)$ of closed subsets of $F(X)$ containing exactly one element. It is therefore enough to show that $F_1(X)$ is a Borel subset of $F(X)$. Fix a countable basis $\mathcal{A}$ of open Polish subsets of X. Suppose also that for every $U \in \mathcal{A}$ it is fixed a compatible complete metric d_U on U. Observe that $F_1(X)$ contains precisely the closed subsets F of X such that $F \cap U \neq \varnothing$ for some $U \in \mathcal{A}$ and for every $U \in \mathcal{A}$ such that $F \cap U \neq \varnothing$ and for every $n \in \omega$ there is $W \in \mathcal{A}$ such that $cl(W) \subset U$, $diam_U\left(cl\left(\overline{W}\right)\right) < 2^{-n}$ and $F \cap (X \backslash cl(W)) = \varnothing$, where $diam_U\left(cl\left(\overline{W}\right)\right)$ is the diameter of $\overline{W}$ with respect to the metric d_U. This shows that $F_1(X)$ is a Borel subset of X.

2.3. **The Effros fibred space.** Suppose that Z is a locally Polish space, X is a Polish space, and $p : Z \to X$ is a continuous open surjection. For $x \in X$ denote by Z_x the inverse image of x under p. Define $F^*(Z)$ to be the space of *nonempty* subsets of Z endowed with the Effros Borel structure. Define $F^*(Z, X)$ to be the Borel subset of closed subsets of Z contained in Z_x for some $x \in X$. The Borel function from $F^*(Z, X)$ onto X assigning to an element F of $F^*(Z, X)$ the unique $x \in X$ such that $F \subset Z_x$ endows $F^*(Z, X)$ with the structure of fibred Borel space. The obvious embedding of $F^*(Z_x)$ into $F^*(Z, X)$ is a Borel isomorphism onto the x-fiber of $F^*(Z, X)$.

Consider the set $\{\varnothing_x : x \in X\}$ endowed with the Borel structure obtained from the bijection $x \leftrightarrow \varnothing_x$. Define $F(Z, X)$ to be the disjoint union of $F^*(Z, X)$ with $\{\varnothing_x : x \in X\}$, which is again fibred over X in the obvious way. Moreover the x-fiber of $F(Z, X)$ is now naturally isomorphic to the space $F(Z)$ of (possibly empty) subsets of Z_x. We will call $F(Z, X)$ the (standard) *Effros fibred space* of the fibration $p : Z \to X$.

2.4. **Analytic and Borel groupoids.** A *groupoid* G is a small category where every arrow is invertible. The set of objects of G is denoted by G^0. We will regard G^0 as a subset of G, by identifying an object with its identity arrow. Denote by G^2 the (closed) set of pairs of composable arrows

$$G^2 = \{(\gamma, \rho) : s(\gamma) = r(\rho)\}.$$

If A, B are subsets of G, then AB stands for the set

$$\left\{\gamma\rho : (\gamma, \rho) \in (A \times B) \cap G^2\right\}.$$

In particular if $Y \subset X$ then

$$YB = \{\gamma \in B : r(\gamma) \in Y\}$$

and

$$BY = \{\gamma \in B : s(\gamma) \in Y\}.$$

We write xB for $\{x\} B = r^{-1}[\{x\}] \cap B$ and Bx for $B\{x\} = s^{-1}[\{x\}] \cap B$. If A is a set of objects, then the *restriction* $G_{|A}$ of G to A (this is called "contraction" in [36, 41]) is the groupoid

$$\{\gamma \in G : s(\gamma) \in A, r(\gamma) \in A\}$$

with set of objects A and operations inherited from G.

To every groupoid G one can associate the *orbit equivalence relation* E_G on G^0 defined by $(x, y) \in E_G$ if and only if there is $\gamma \in G$ such that $s(\gamma) = x$ and $r(\gamma) = y$. The function

$$\begin{array}{rcl} G & \to & E_G \\ \gamma & \mapsto & (r(\gamma), s(\gamma)) \end{array}$$

is a continuous surjection. We say that a groupoid is *principal* when such a map is injective. Thus a principal groupoid is just an equivalence relation on its set of objects. Conversely any equivalence relation can be regarded as a principal groupoid.

The notion of *functor* between groupoids is the usual notion from category theory. Thus a functor from G to H is a function from G to H such that, for every $\gamma \in G$ and $(\rho_0, \rho_1) \in G^2$ the following holds:

- $F(s(\gamma)) = s(F(\gamma))$;
- $F(r(\gamma)) = r(F(\gamma))$;
- $F(\gamma^{-1}) = F(\gamma)^{-1}$;
- $F(\rho_0\rho_1) = F(\rho_0)F(\rho_1)$.

When E and E' are principal groupoids, then functors from E to E' are in 1:1 correspondence with *reductions* from E to E' in the sense of [18, Definition 5.1.1].

Definition 2.4.1. An *analytic groupoid* is a groupoid endowed with an analytic Borel structure making composition and inversion of arrows Borel and such that the set of objects and, for every object x, the set of elements with source x, are standard Borel spaces with respect to the induced Borel structure. A *(standard) Borel groupoid* is a groupoid endowed with a standard Borel structure making composition and inversion of arrows Borel, and such that the set of objects is a Borel subset.

It is immediate to verify that principal analytic groupoids are precisely analytic equivalence relations on standard Borel spaces. Similarly principal Borel groupoids are precisely the Borel equivalence relations on standard Borel spaces. A functor between analytic groupoids is Borel if it is Borel as a function with respect to the given Borel structures.

2.5. **Polish groupoids and Polish groupoid actions.**

Definition 2.5.1. A *topological groupoid* is a groupoid endowed with a topology making composition and inversion of arrows continuous.

It is not difficult to see that for a topological groupoid the following conditions are equivalent:

(1) Composition of arrows is open;
(2) The source map is open;
(3) The range map is open.

(See [44, Exercise I.1.8].)

Definition 2.5.2. A *Polish groupoid* is a groupoid endowed with a locally Polish topology such that

(1) composition and inversion of arrows are continuous and open,
(2) the set G^0 of objects is a Polish space with the subspace topology,
(3) for every $x \in G^0$ the sets Gx and xG are Polish spaces with the subspace topology.

Polish groupoids have been introduced in [42] with the extra assumption that the topology be regular or, equivalently, globally Polish. It is nonetheless noticed in [42, page 362] that one can safely dispense of this additional assumption, without invalidating the results proved therein.

Suppose that G is a Polish groupoid, and X is a Polish space. A continuous action of G on X is given by a continuous function $p : X \to G^0$ called *anchor map* together with a continuous function $(g, x) \mapsto gx$ from

$$G \ltimes X = \{(\gamma, x) : p(x) = s(\gamma)\}$$

to X such that, for all $\gamma, \rho \in G$ and $x \in X$

(1) $\gamma(\rho x) = (\gamma\rho)x$,
(2) $p(\gamma x) = r(\gamma)$, and
(3) $p(x)x = x$.

In such a case we say that X is a Polish G-space. Similarly if X is a standard Borel space, then a Borel action of G on X is given by a Borel map $p : X \to G^0$ together with a Borel map

$$\begin{array}{rcl} G \ltimes X & \to & X \\ (\gamma, x) & \to & \gamma x \end{array}$$

satisfying the same conditions as above. In this case X will be called a Borel G-space.

Clearly any Polish groupoid acts continuously on its space of objects G^0 by setting $p(x) = x$ and $(\gamma, x) \mapsto r(\gamma)$. This will be called the *standard* action of G on G^0.

Most of the usual notions for actions of groups, such as orbits, or invariant sets, can be generalized in the obvious way to actions of groupoids. If X is a G-space, and $x \in X$, then its orbit $\{\gamma x : s(\gamma) = p(x)\}$ is denoted by $[x]$. The orbit equivalence relation E_G^X on X is defined by $xE_G^X y$ iff $[x] = [y]$. If A is a subset of X, then its saturation

$$\{\gamma a : a \in A, \, \gamma \in Gp(a)\}$$

is denote by $[A]$. An action is called *free* if $\gamma x = \rho x$ implies $\gamma = \rho$ for any $x \in X$ and $\gamma, \rho \in Gp(x)$.

Suppose that G is a Polish groupoid, and X is a Borel G-space. If $x, y \in G^0$ are in the same orbit define the stabilizer

$$G_x = \{\gamma \in G : s(\gamma) = p(x) \text{ and } \gamma x = x\}$$

of x, and

$$G_{x,y} = \{\gamma \in G : s(\gamma) = p(x) \text{ and } \gamma x = y\}.$$

Observe that by [28, Theorem 9.17] G_x is a closed subgroup of $p(x)Gp(x)$. Therefore $G_{x,y}$ is also closed, since $G_{x,y} = G_{x,x}\rho$ for any ρ such that $s(\rho) = p(x)$ and $\rho x = y$.

Suppose that X and Y are Borel G-spaces with anchor maps p_X and p_Y. A Borel fibred map from X to Y is a Borel function $\varphi : X \to Y$ such that $p_Y \circ \varphi = p_X$. A Borel fibred map from X to Y is *G-equivariant* if

$$\varphi(\gamma x) = \gamma\varphi(x)$$

for $x \in X$ and $\gamma \in Gp(x)$. A *Borel G-embedding* from X to Y is an injective G-equivariant Borel fibred map from X to Y. Finally a *Borel G-isomorphism* from X to Y is a Borel G-embedding which is also onto.

2.6. **Some examples of Borel groupoids.** In this subsection we show how several natural categories of interest can be endowed (after a suitable parametrization) with the structure of Borel groupoid.

Let us first consider the category of *complete separable metric spaces*, having surjective isometries as morphisms. This can be endowed with the structure of Borel groupoid in the following way. Denote by $\mathbb{U}$ the Urysohn universal metric space. (A survey about $\mathbb{U}$ and its remarkable properties can be found in [38].) Let $F(\mathbb{U})$ be the Borel space of closed subsets of $\mathbb{U}$ endowed with the Effros Borel structure. By universality of the Urysohn space, $F(\mathbb{U})$ contains an isometric copy of any separable metric space. Moreover any surjective isometry between closed subsets of $\mathbb{U}$ can be identified with its graph, which is a closed subset of $\mathbb{U} \times \mathbb{U}$. The set **CMS** of such graphs is easily seen to be a Borel subset of $F(\mathbb{U})$. Moreover a standard computation shows that composition and inversion of arrows are Borel functions in **CMS**. This shows that **CMS** is a Borel groupoid that can be seen as a parametrization of the category of metric spaces with surjective isometries as arrows.

More generally one can look at the category of *separable $\mathcal{L}$-structures* in some signature $\mathcal{L}$ of continuous logic. (A complete introduction to continuous logic is [4].) One can identify any $\mathcal{L}$-structure with an $\mathcal{L}$-structure having as support a closed subset of $\mathbb{U}$. In such case the interpretation of a function symbol f can be seen as a closed subset of $\mathbb{U}^{|f|+1}$ where $|f|$ denotes the arity of f. The interpretation of a relation symbol B can be seen as a closed subset of $\mathbb{U}^{|R|} \times \mathbb{R}$ where again $|R|$ denotes the arity of B. (Here distances and relations are allowed to attain value in the whole real line.)

The set $\mathrm{Mod}(\mathcal{L})$ of such structures can be verified to be a Borel subset of

$$F(\mathbb{U}) \times \prod_f F(\mathbb{U}) \times \prod_B F(\mathbb{U})$$

where f and B range over the function and relation symbols of $\mathcal{L}$. Similar parametrizations of the space of $\mathcal{L}$-structures can be found in [9] and [5]. As before the space $\mathrm{Mod}(\mathcal{L})$ of isomorphisms between $\mathcal{L}$-structures (identified with their graph) is a Borel subset of $F(\mathbb{U})$, and composition and inversion of arrows are Borel maps. Thus one can regard $\mathrm{Mod}(\mathcal{L})$ as the Borel groupoid of $\mathcal{L}$-structures. In the particular case when one considers *discrete* structures then one can replace the Urysohn space with ω.

As a particular case of separable structures in a given signature one can consider *separable C*-algebras*. (The book [6] is a complete reference for the theory of operator algebras.) The complexity of the classification problem for separable C*-algebras has recently attracted considerable interest; see [9, 13, 14, 45]. Particularly important classes for the classification program are *nuclear* and *exact* C*-algebras; see [6, Section IV.3]. Separable exact C*-algebras are precisely the closed self-adjoint subalgebras of the Cuntz algebra $\mathcal{O}_2$ [32]. Thus the Borel groupoid $\mathbf{C^*Exact}$ of closed subalgebras of $\mathcal{O}_2$—where a *-isomorphism between closed subalgebras is identified with its graph—can be regarded as a parametrization for the category of exact C*-algebras having *-isomorphisms as arrows. The category of (simple, unital) nuclear C*-algebras can be regarded as the restriction of $\mathbf{C^*Exact}$ to the Borel set of (simple, unital) self-adjoint subalgebras of $\mathcal{O}_2$; see [14, Section 7].

We now look at the category of *Polish groups* with continuous group isomorphisms as arrows. Denote by $\mathrm{Iso}(\mathbb{U})$ the group of isometries of the Urysohn space endowed with the topology of pointwise convergence. Recall that $\mathrm{Iso}(\mathbb{U})$ is a universal Polish group [47], i.e. it contains any other Polish group as closed subgroup. The space $SG(\mathrm{Iso}(\mathbb{U}))$ of closed subgroups of $\mathrm{Iso}(\mathbb{U})$ endowed with the Effros Borel structure can be regarded as the standard Borel space of Polish groups. Moreover a continuous isomorphism between closed subgroups of $\mathrm{Iso}(\mathbb{U})$ can be identified with its graph, which is a closed subgroup of $\mathrm{Iso}(\mathbb{U}) \times \mathrm{Iso}(\mathbb{U})$. It is not difficult to check that the set $\mathbf{PG}$ of such closed subgroups of $\mathrm{Iso}(\mathbb{U}) \times \mathrm{Iso}(\mathbb{U})$ is a Borel subset of the space $SG(\mathrm{Iso}(\mathbb{U}) \times \mathrm{Iso}(\mathbb{U}))$ of closed subgroups of $\mathrm{Iso}(\mathbb{U}) \times \mathrm{Iso}(\mathbb{U})$ endowed with the Effros Borel structure. (Fix a countable neighborhood basis $\mathcal{N}$ of the identity in $\mathrm{Iso}(\mathbb{U})$, and observe that a closed subgroup H of $\mathrm{Iso}(\mathbb{U}) \times \mathrm{Iso}(\mathbb{U})$ is in $\mathbf{PG}$ if and only if $\forall U \in \mathcal{N}\ \exists V \in \mathcal{N}$ such that $H \cap (U \times (\mathrm{Iso}(\mathbb{U}) \backslash cl(V))) = \varnothing$ and $H \cap ((\mathrm{Iso}(\mathbb{U}) \backslash cl(V)) \times U) = \varnothing$.) Moreover a standard calculation shows that composition and inversion of arrows are Borel functions in $\mathbf{PG}$. (For composition of arrows, observe that if as before $\mathcal{N}$ is a countable basis of neighborhoods of the identity in $\mathrm{Iso}(\mathbb{U})$, D is a dense subset of $\mathrm{Iso}(\mathbb{U})$, A and B are open subsets of $\mathrm{Iso}(\mathbb{U})$, and $\varphi, \psi \in \mathbf{PG}$, then $(\varphi \circ \psi) \cap (A \times B) \neq \varnothing$ if and only if there are $U, V \in \mathcal{N}$ and $g, h \in D$

with $cl(V)^2h \subset B$ and $Ug \subset A$ such that $\psi \cap \left(U^2 \times (\mathrm{Iso}(\mathbb{U})\backslash cl(V))\right) = \varnothing$, $\varphi \cap (A \times Ug) \neq \varnothing$, and $\psi \cap (Ug \times Vh) \neq \varnothing$.) This shows that **PG** is a Borel groupoid that can be seen as a parametrization of the category of Polish groups with continuous isomorphisms as arrows.

A similar discussion applies to the category of separable Banach spaces with linear (not necessarily isometric) isomorphisms as arrows. In this case one considers a universal separable Banach space, such as $C[0,1]$. One then looks at the standard Borel space of closed subspaces of $C[0,1]$ as set of objects, and the set of closed subspaces of $C[0,1] \oplus C[0,1]$ that code a linear isomorphism between closed subspaces of $C[0,1]$ as set of arrows. The proof that these sets are Borel with respect to the Effros Borel structure is analogous to the case of Polish groups.

2.7. **The action groupoid.** Suppose that G is a Polish groupoid, and X is a Polish G-space. Consider the groupoid

$$G \ltimes X = \{(\gamma, x) \in G \times X : s(\gamma) = p(x)\},$$

where composition and inversion of arrows are defined by

$$(\rho, \gamma x)(\gamma, x) = (\rho\gamma, x)$$

and

$$(\gamma, x)^{-1} = \left(\gamma^{-1}, \gamma x\right).$$

The set of objects of $G \ltimes X$ is

$$G^0 \ltimes X = \left\{(a, x) \in G^0 \times X : p(x) = a\right\}.$$

Endow $G \ltimes X$ with the subspace topology from $G \times X$. Observe that the function

$$\begin{array}{rcl} X & \to & G^0 \ltimes X \\ x & \mapsto & (p(x), x) \end{array}$$

is a homeomorphism from X to the set of objects of $G \ltimes X$. We can therefore identify the latter with X. Under this identification the source of (γ, x) is x and the range is γx. We claim that $G \ltimes X$ is a Polish groupoid, called the *action groupoid* associated with the Polish G-space X. Clearly the topology is locally Polish, and composition and inversion of arrows are continuous. We need to show that the source map is open. Suppose that V is an open subset of G, U is an open subset of X, and W is the open subset

$$\{(\gamma, x) : \gamma \in V, x \in U\}$$

of $G \ltimes X$. Suppose that W is nonempty and pick $(\gamma_0, x_0) \in W$. Thus $x_0 \in U$ and $p(x_0) = s(\gamma_0) \in s[V]$. Therefore there is an open subset U_0 of U containing x_0 such that $p[U_0] \subset s[V]$. We claim now that U_0 is contained in the image of W under the source map. In fact if $x \in U_0$ then $p(x) = s(\gamma)$ for some $\gamma \in V$ and therefore x is the source of the arrow (γ, x) in W. This concludes the proof of the fact that $G \ltimes X$ is a Polish groupoid. To summarize we can state the following proposition.

Proposition 2.7.1. *Suppose that G is a Polish groupoid, and X is a Polish G-space. The action groupoid $G \ltimes X$ as defined above is a Polish groupoid. Moreover the map*

$$\begin{array}{rcl} X & \to & (G \ltimes X)^0 \\ x & \mapsto & (p(x), x) \end{array}$$

is a homeomorphism such that, for every $x, x' \in X$,

$$x E_G^X x' \quad \textit{iff} \quad (p(x), x)\, E_{G \ltimes X} \left(p(x'), x'\right).$$

2.8. **Functorial reducibility.**

Definition 2.8.1. Suppose that G and H are analytic groupoids. A *Borel reduction* from G to H is a Borel functor F from G to H such that $xGy \neq \varnothing$ whenever $F(x)HF(y) \neq \varnothing$.

Equivalently a Borel functor F from G to H is a Borel reduction from G to H when the function

$$\begin{array}{rcl} G^0 & \to & H^0 \\ x & \mapsto & F(x) \end{array}$$

is a Borel reduction from E_G to E_H in the sense of [18, Definition 5.1.1].

Definition 2.8.2. Suppose that G and H are analytic groupoids. We say that G is *Borel reducible* to H—in formulas $G \leq_B H$—if there is a Borel reduction from G to H.

The notion of bireducibility is defined accordingly.

Definition 2.8.3. Suppose that G and H are analytic groupoids. We say that G is *Borel bireducible* to H—in formulas $G \sim_B H$—if G is Borel reducible to H and vice versa.

When E and E' are principal analytic groupoids, then the Borel reductions from E to E' are in 1:1 correspondence with Borel reductions from E to E' in the usual sense of Borel complexity theory; see [18, Definition 5.1.1]. In particular Definition 2.8.2 *generalizes* the notion of Borel reducibility from analytic equivalence relations to analytic groupoids.

Similarly as in the case of reducibility for equivalence relations, one can impose further requirements on the reduction map. If G and H are analytic groupoid, we say that G is injectively Borel reducible to H—in formulas $G \sqsubseteq_B H$ if there is an *injective* Borel reduction from G to H. When G and H are Polish groupoid, one can also insist that the reduction be continuous rather than Borel. One then obtains the notion of continuous reducibility $\leq_c$ and continuous injective reducibility $\sqsubseteq_c$.

Definition 2.8.2 provides a natural notion of comparison between analytic groupoids. This allows one to build a hierarchy of complexity of analytic groupoids, that includes the usual hierarchy of Borel equivalence relations. The *functorial Borel complexity* of an analytic groupoid will denote the position of the given groupoid in such a hierarchy.

2.9. **Category preserving maps.** According to [39, Definition A.2] a continuous map $f : X \to Y$ between Polish spaces is *category preserving* if for any comeager subset C of Y the inverse image $f^{-1}[C]$ of C under f is a comeager subset of X. It is not difficult to see that any continuous open map is category preserving [39, Proposition A.3].

Category-preserving maps satisfy a suitable version of the classical Kuratowski-Ulam theorem for coordinate projections. We will state the particular case of this result for continuous open maps in the following lemma, which is Theorem A.1 in [39].

Lemma 2.9.1. *Suppose that X is second countable space, Y is a Baire space, and $f : X \to Y$ is an open continuous map such that $f^{-1}\{y\}$ is a Baire space for every $y \in Y$. If $A \subset X$ has the Baire property, then the following statements are equivalent:*

1. *A is comeager;*
2. *$\forall^* y \in Y$, $A \cap f^{-1}\{y\}$ is comeager in $f^{-1}\{y\}$.*

2.10. **Vaught transforms.** Suppose in the following that G is a Polish groupoid,

$$\mathcal{A} = \{U_n : n \in \omega\}$$

is a basis of Polish open subsets of G, and X is a Borel G-space.

Definition 2.10.1. For $A \subset X$ and $V \subset G$, define the *Vaught transforms*

$$A^{\triangle V} = \{x \in X : Vp(x) \neq \varnothing \text{ and } \exists^* \gamma \in Vp(x),\, \gamma x \in A\}$$

and

$$A^{*V} = \{x \in X : Vp(x) \neq \varnothing \text{ and } \forall^* \gamma \in Vp(x),\, \gamma x \in A\}.$$

In the particular case when G is a Polish group, and X is a Borel G-space, this definition coincide with the usual Vaught transform; cf. [18, Definition 3.2.2].

Lemma 2.10.2. *Assume that B and A_n for $n \in \omega$ are subsets of X. If V is an open subset of G, then the following hold:*

1. *$B^{\triangle G}$ and B^{*G} are invariant subsets of X;*
2. *$\left(\bigcap_n A_n\right)^{*V} = \bigcap_n A_n^{*V}$;*
3. *$\left(\bigcup_n A_n\right)^{\triangle V} = \bigcup_n A_n^{\triangle V}$;*
4. *$p^{-1}[s[V]]$ is the disjoint union of $(X \setminus B)^{*V}$ and $B^{\triangle V}$;*
5. *If B is analytic, then $B^{\triangle V} = \bigcup \left\{B^{*U} : V \supset U \in \mathcal{A}\right\}$ and $B^{*V} = \bigcap \left\{B^{\triangle U} : V \supset U \in \mathcal{A}\right\}$.*

Lemma 2.10.2 is elementary and can be proved similarly as [18, Proposition 3.2.5].

Lemma 2.10.3. *Suppose that $B \subset X$ is analytic, and $U \subset G$ is open. If $x \in X$ and $\gamma \in Gp(x)$, then the following statements are equivalent:*

1. *$\gamma x \in B^{\triangle U}$;*
2. *$x \in B^{*V}$ for some $V \in \mathcal{A}$ such that $V\gamma^{-1} \subset Ur(\gamma)$;*

(3) *$x \in B^{\triangle V}$ for some $V \in \mathcal{A}$ such that $V\gamma^{-1} \subset Ur(\gamma)$;*
(4) *there are $V, W \in \mathcal{A}$ such that $VW^{-1} \subset U$, $\gamma \in W$, and $x \in B^{\triangle V}$.*

(1)⇒(2): By hypothesis $Ur(\gamma) \neq \varnothing$ and $\exists^* \rho \in Ur(\gamma)$ such that $\rho\gamma x \in B$. Therefore $U\gamma \neq \varnothing$ and $\exists^* \rho \in U\gamma$ such that $\rho x \in B$. Since B is analytic and the action is Borel, the set

$$\{\rho \in U\gamma : \rho x \in B\}$$

is analytic and in particular it has the Baire property. It follows that there is $V \in \mathcal{A}$ such that $Vp(x) \neq \varnothing$, $Vp(x) \subset U\gamma$, and $\forall^* \rho \in V$, $\rho x \in B$. Observe that $V\gamma^{-1} \subset Ur(\gamma)$.

(2)⇒(3): Obvious.

(3)⇒(1): Observe that $\varnothing \neq Vp(x) \subset U\gamma$. Thus $U\gamma \neq \varnothing$ and $\exists^* \rho \in U\gamma$ such that $\rho x \in B$. Thus $Up(\gamma z) \neq \varnothing$ and $\exists^* \rho \in Up(\gamma x)$, $\rho\gamma x \in B$. This shows that $\gamma x \in B^{\triangle U}$.

(2)⇒(4): Pick $v \in Vp(x)$ and observe that $v\gamma^{-1} \in Ur(\gamma)$. Therefore there are $W, V_0 \in \mathcal{A}$ such that $v \in V_0 \subset V$, $\gamma \in W$, and $V_0 W^{-1} \subset U$. Moreover since $x \in B^{*V}$ and $V_0 \subset V$ we have that $x \in B^{*V_0}$.

(4)⇒(2): Obvious.

If A is a subset of $G \ltimes X$ and $x \in X$, then A_x denotes the x-fiber $\{\gamma \in G : (\gamma, x) \in A\}$ of A. The proof of the following lemma is inspired by the proof of the Montgomery-Novikov theorem; see [28, Theorem 16.1].

Lemma 2.10.4. *If A is a Borel subset of $G \ltimes X$ and $V \subset G$ is open, then*

$$\{x \in X : Vp(x) \neq \varnothing \text{ and } A_x \text{ is nonmeager in } Vp(x)\}$$

is Borel. The same conclusion holds if one replaces "nonmeager" with "comeager" or "meager".

Proof. Define $\mathcal{E}$ to be the class subsets of subsets A of $G \ltimes X$ such that

$$\{x \in X : Vp(x) \neq \varnothing \text{ and } A_x \text{ is nonmeager in } Vp(x)\}$$

is Borel for every nonempty open subset V of G. We claim that:

(1) $\mathcal{E}$ contains the sets of the form

$$U \ltimes B = \{(\rho, x) \in G \ltimes X : x \in B, \rho \in U\}$$

for $B \subset X$ Borel and $U \subset G$ open;
(2) $\mathcal{E}$ is closed by taking countable unions;
(3) $\mathcal{E}$ is closed by taking complements.

In fact:

(1) If $A = U \ltimes B$ where $B \subset X$ is Borel and $U \subset G$ is open then for every nonempty open set V

$$\begin{aligned} & \{x \in X : Vp(x) \neq \varnothing \text{ and } A_x \text{ is nonmeager in } Vp(x)\} \\ = \quad & B \cap p^{-1}\left[s\left[U \cap V\right]\right]; \end{aligned}$$

is Borel.

(2) If $A=\bigcup_n A_n$ then for every nonempty open set

$$\begin{aligned} & \{x\in X : Vp(x)\neq\varnothing \text{ and } A_x \text{ is nonmeager in } Vp(x)\} \\ = & \bigcup_{n\in\omega}\{x\in X : Vp(x)\neq\varnothing \text{ and } \left(A_n\right)_x \text{ is nonmeager in } Vp(x)\}; \end{aligned}$$

(3) If $A\subset G\ltimes X$ then for every nonempty open set V

$$\begin{aligned} & \{x\in X : Vp(x)\neq\varnothing \text{ and } \left((G\ltimes X)\setminus A\right)_x \text{ is nonmeager in } V\} \\ = & \{x\in X : Vp(x)\neq\varnothing \text{ and } A_x \text{ is not comeager in } V\} \\ = & \bigcup_{U_n\subset V}\{x\in X : U_np(x)\neq\varnothing \text{ and } A_x \text{ is meager in } U_n\} \\ = & \bigcup_{U_n\subset V}\left(p^{-1}s\left[U_n\right]\setminus\{x\in X : U_np(x)\neq\varnothing \text{ and } A_x \text{ is nonmeager in } U_n\}\right). \end{aligned}$$

A similar argument shows that the same conclusion holds after replacing "nonmeager" with "meager" or "comeager". ☐

Lemma 2.10.5. *If $A\subset X$ is Borel and $V\subset G$ is open then $A^{\triangle V}$ and A^{*V} are Borel.*

Proof. Consider the subset

$$\widetilde{A}=\{(\rho,x)\in G\ltimes X : \rho x\in A\}$$

and observe that $\widetilde{A}$ is a Borel subset of $G\ltimes X$ such that

$$A^{\triangle V}=\left\{x\in X : Vp(x)\neq\varnothing \text{ and } \widetilde{A}_x \text{ is nonmeger in } Vp(x)\right\}$$

and

$$A^{*V}=\left\{x\in X : Vp(x)\neq\varnothing \text{ and } \widetilde{A}_x \text{ is comeager in } Vp(x)\right\}.$$

The conclusion now follows from Lemma 2.10.4. ☐

Lemma 2.10.6. *Assume that X is a Polish G-space. If $B\subset X$, $U\subset G$ is open, and $\alpha\in\omega_1$, then the following hold:*

1. *If B is open, then $B^{\triangle U}$ is open;*
2. *If B is $\boldsymbol{\Sigma}^0_\alpha$, then $B^{\triangle U}$ is $\boldsymbol{\Sigma}^0_\alpha$ relatively to $p^{-1}s\left[U\right]$;*
3. *If B is $\boldsymbol{\Pi}^0_\alpha$, then B^{*U} is $\boldsymbol{\Pi}^0_\alpha$ relatively to $p^{-1}s\left[U\right]$.*

Proof. The proof is analogous to the corresponding one for group actions; see [18, Theorem 3.2.9]. Suppose that B is open, and pick $x\in B^{\triangle U}$. Thus $Up(x)\neq\varnothing$ and $\exists^*\rho\in Up(x)$ such that $\rho x\in B$. Pick $U_0\subset U$ open such that $x\in B^{*U_0}$ and $\rho\in U_0p(x)$ such that $\rho x\in B$. Since B is open and the action is continuous there are open subsets W and V containing x and ρ such that $V\subset U_0$, $VW\subset B$, and $p\left[W\right]\subset s\left[V\right]$. We claim that $W\subset B^{\triangle U}$. In fact if $w\in W$ then $Vp(w)\neq\varnothing$. Moreover since $VW\subset B$, $\exists^*\rho\in Vp(w)$ such that $\rho w\in B$. This concludes the proof that $B^{\triangle U}$ is open. The other statements follow via (2),(3), and (4) of Lemma 2.10.2. ☐

Using the Vaught transform it is easy to see that, if X is a Borel G-space, then the orbit equivalence relation E_G^X is idealistic. (This is well known when G is a Polish group; cf. [18, Proposition 5.4.10].) Recall that an equivalence relation E on a standard Borel space X is *idealistic* if there is a map $[x]_E \mapsto I_{[x]_E}$ assigning to each equivalence class $[x]_E$ of E and ideal $I_{[x]_E}$ of subsets of $[x]_E$ such that $[x]_E \notin I_{[x]_E}$, and for every Borel subset A of $X \times X$ the set A_I defined by $x \in A_I$ iff $\{y \in [x]_E : (x,y) \in A\} \in I_{[x]_E}$ is Borel; see [18, Definition 5.4.9].

Proposition 2.10.7. *If X is a Borel G-space, then the orbit equivalence relation E_G^X is idealistic.*

Proof. Pick $x \in X$ and denote by C the orbit of x. Define the ideal I_C of subsets of C by $S \in I_C$ iff $\forall^* \rho \in Gp(x)$, $r(\rho) \notin S$. Observe that this does not depend from the choice of x. In fact suppose that $y \in C$ and hence $y = \gamma x$ for some $\gamma \in Gp(x)$. Assume moreover that $S \subset C$ is such that $\forall^* \rho \in Gp(x)$, $r(\rho) \notin S$. Consider the homeomorphism Φ from $Gp(x)$ to $Gp(y)$ given by $\rho \mapsto \rho\gamma$. It is apparent that

$$\Phi\left[\{\rho \in Gp(x) : r(\rho) \notin S\}\right] = \{\rho \in Gp(y) : r(\rho) \notin S\}.$$

This shows that $\forall^* \rho \in Gp(y)$, $r(\rho) \notin S$, and hence the definition of I_C is does not depend from the choice of $x \in C$. Clearly $C \notin I_C$ since $Gp(x)$ is a Baire space. It is not difficult to verify that I_C is a σ-ideal. Suppose that $A \subset X \times X$ is Borel, and consider the set A_I defined by $x \in A_I$ iff $\{y \in [x] : (x,y) \in A\} \in I_{[x]}$. Observe that $x \in A_I$ iff $\forall^* \rho \in Gp(x)$, $(x, r(\rho)) \notin A$. Consider

$$X * X = \{(x,y) \in X \times X : p(x) = p(y)\}$$

and the action of G on $X * X$ defined by $p(x,y) = p(x) = p(y)$ and $\gamma(x,y) = (x, \gamma y)$ for $\gamma \in Gp(x,y)$. Observe that $x \in A_I$ if and only if $(x,x) \in ((X * X) \setminus A)^{*G}$. This shows that A_I is Borel by Lemma 2.10.5. $\square$

Let us denote as customary by E_1 the tail equivalence relation for sequences in $[0,1]$. If E is an idealistic Borel equivalence relation, then E_1 is not Borel reducible to E by [26, Theorem 4.1]. Therefore we obtain from Proposition 2.10.7 the following corollary:

Corollary 2.10.8. *If X is a Borel G-space with Borel orbit equivalence relations, then the orbit equivalence relation E_1 is not Borel reducible to E_G^X.*

Corollary 2.10.8 holds more generally for arbitrary Borel G-spaces, with not necessarily Borel orbit equivalence relations. This was shown by the present author in collaboration with Samuel Coskey, George Elliott, and Ilijas Farah by adapting the proof of [21, Chapter 8].

An equivalence relation E on a standard Borel space E is *smooth* if it is Borel reducible to the relation of equality in some Polish space [18, Definition 5.4.1]. By [18, Theorem 5.4.11] an equivalence relation has a Borel selector precisely when it is smooth and idealistic. Therefore the following corollary follows immediately from Proposition 2.10.7.

Corollary 2.10.9. *If X is a Polish G-space such that E_G^X is smooth, then E_G^X has a Borel selector.*

Corollary 2.10.10. *If G and H are Polish groupoids such that E_G and E_H are smooth, then $G \leq_B H$ if and only if $E_G \leq E_H$.*

2.11. **Borel orbits.** We now observe that, if G is a Polish groupoid, then the orbits of any Polish G-space are Borel.

Proposition 2.11.1. *If G is a Polish groupoid, and X is a Polish G-space, then the orbit equivalence relation E_G^X is analytic and has Borel classes.*

Proof. By Proposition 2.7.1 we can consider without loss of generality the case of the standard action of G on its set of objects G^0. Fix $x \in G^0$ and consider the right action of xGx on Gx by composition. Observe that xGx is a Polish group, and Gx is a right Polish xGx-space with closed orbits. Therefore by [18, Proposition 3.4.6] the corresponding orbit equivalence relation E_{xGx}^{Gx} has a Borel transversal T. The orbit $[x]$ is the image of T under the range map r. Since r is 1:1 on T, it follows that $[x]$ is Borel by [28, Theorem 15.1]. Observe now that the orbit equivalence relation E_G is the image of the standard Borel space G under the Borel function $\gamma \mapsto (r(\gamma), s(\gamma))$. This shows that E_G is analytic. $\square$

Similarly as in the case of Polish group actions, a uniform bound on the complexity of the orbits in the Borel hierarchy entails Borelness of the orbit equivalence relation.

Theorem 2.11.2. *Suppose that G is Polish groupoid, and X is a Polish G-space. The orbit equivalence relation E_G^X is Borel if and only if there is $\alpha \in \omega_1$ such that every orbit is $\mathbf{\Pi}_\alpha^0$*

Proof. One direction is obvious. For the other one consider for $\alpha \in \omega_1$ the relation E_α of X defined by

$$(x, y) \in E_\alpha$$

iff for every G-invariant $\mathbf{\Pi}_\alpha^0$ set $W \subset X$ we have that $x \in W$ iff $y \in W$. If every orbit is $\mathbf{\Pi}_\alpha^0$ then $E_G^X = E_\alpha$. It is thus enough to prove that E_α is co-analytic for every $\alpha \in \omega_1$. Consider a universal $\mathbf{\Pi}_\alpha^0$ subset U of $\omega^\omega \times X$. Define the action of G on $\omega^\omega \times X$ by setting $p\,(a, b) = p(b)$ and $\gamma\,(a, b) = (a, \gamma b)$. Define now

$$T = U^{*G}$$

and observe that T is $\mathbf{\Pi}_\alpha^0$ since U is $\mathbf{\Pi}_\alpha^0$. Denote by T_a the section

$$\{b \in X : (a, b) \in T\}$$

for $a \in \omega^\omega$. We have that

$$\begin{aligned} b \in T_a &\Leftrightarrow (a,b) \in T \\ &\Leftrightarrow \forall^* \gamma \in Gp(b),\ (a, \gamma b) \in U \\ &\Leftrightarrow \forall^* \gamma \in Gp(b), \gamma b \in U_a \\ &\Leftrightarrow b \in (U_a)^{*G}. \end{aligned}$$

This shows that T_a is a G-invariant $\mathbf{\Pi}^0_\alpha$ subset of X for every $\alpha \in \omega^\omega$. Conversely if A is a G-invariant $\mathbf{\Pi}^0_\alpha$ set then $A = U_a$ for some $a \in \omega^\omega$ and hence

$$A = A^{*G} = (U_a)^{*G} = T_a.$$

This shows that $\{T_a : a \in \omega^\omega\}$ is the collection of all invariant $\mathbf{\Pi}^0_\alpha$ sets. It follows that $(x,y) \in E_\alpha$ iff $\forall a \in \omega^\omega$, $(a,x) \in T$. Therefore E_α is co-analytic. □

Theorem 2.11.2 was proved for Polish group actions in [46, Sections 3.6 and 3.7].

3. Effros' theorem and the Glimm-Effros dichotomy

3.1. **Effros' theorem.**

Lemma 3.1.1. *Suppose that G is a Polish groupoid. Consider the standard action of G on G^0, and the corresponding Vaught transform. If $A \subset G^0$ is meager, then $A^{\triangle G}$ is meager.*

Proof. The source map $r : G \to G^0$ is open and, in particular, category preserving; see Subsection 2.9. Thus $r^{-1}[A]$ is a meager subset of G. Therefore, since the source map s is also open, by Lemma 2.9.1 the set of $x \in X$ such that $Gx \cap r^{-1}[A]$ is meager. This set is by definition $A^{\triangle G}$. □

Theorem 3.1.2. *Suppose that G is a Polish groupoid, X is a Polish G-space, and $x \in X$. Denote by $[x]$ the orbit of x. The following statements are equivalent:*

(1) $[x]$ is a G_δ subset of X;
(2) $[x]$ is a Baire space;
(3) $[x]$ is nonmeager in itself.

Proof. By Proposition 2.7.1 we can assume without loss of generality that $X = G^0$ and $G \curvearrowright G^0$ is the standard action. The only nontrivial implication is $3 \Rightarrow 1$. After replacing G with the restriction of G to the closure of $[x]$, we can assume that $[x]$ is dense in G^0 and hence nonmeager in G^0. By Proposition 2.11.1, $[x]$ is a Borel subset of G^0 and in particular it has the Baire property. Therefore by [28, Proposition 8.23] the orbit $[x]$ is the union of a meager set M and a G_δ set U. One can conclude that $[x] = U^{*G}$ arguing as in [46, Proposition 4.4]. Clearly $[x]$ is the union of U^{*G} and $M^{\triangle G}$. By Lemma 3.1.1, $M^{\triangle G}$ is meager and hence, since $[x]$ is nonmeager, $M^{\triangle G} = \varnothing$. Therefore $[x] = U^{*G}$ is G_δ by Lemma 2.10.6. □

Theorem 2.1 of [42] asserts that it is equivalent for the conditions in Theorem 3.1.2 to hold for all points of X.

Suppose that G is a Polish groupoid, X is a Polish G-space, and $x \in G^0$. The fiber $Gp(x)$ is a Polish space, and the stabilizer G_x of x is a Polish group acting from the right by composition on $Gp(x)$. One can then consider the quotient space $Gp(x)/G_x$ and the quotient map $\pi_x : Gp(x) \to Gp(x)/G_x$, which is clearly continuous and open. When $G \curvearrowright G^0$ is the standard action of G on its set of objects and $x \in G^0$, then the stabilizer G_x is just xGx.

It is not difficult to see that the proof of [42, Theorem 3.2] can be adapted to the context where G is a not necessarily regular Polish groupoid, as observed in [42, page 362]. The following lemma can then be obtained as an immediate consequence.

Lemma 3.1.3 (Ramsay)**.** *Suppose that G is a Polish groupoid, and $x \in G^0$. If the orbit $[x]$ of x is G_δ, then the map $\phi_x : Gx/xGx \to [x]$ defined by $\phi_x(\pi(\gamma)) = r(\gamma)$ is a homeomorphism.*

Corollary 3.1.4. *Suppose that G is a Polish groupoid, X is a Polish G-space, and $x \in X$. If the orbit $[x]$ of x is G_δ, then the map $\phi_x : Gp(x)/G_x \to [x]$ defined by $\phi_x(\pi(\gamma)) = \gamma x$ is a homeomorphism.*

Proof. Consider the action groupoid $G \ltimes X$, and let us identify X with the space of objects of $G \ltimes X$ as in Proposition 2.7.1. Consider the map ψ defined by

$$\begin{aligned} Gp(x) &\to (G \ltimes X)\, x \\ \gamma &\mapsto (\gamma, x). \end{aligned}$$

Observe that ψ is a continuous map with continuous inverse

$$\begin{aligned} (G \ltimes X)\, x &\to Gp(x) \\ (\gamma, x) &\mapsto \gamma. \end{aligned}$$

Moreover the image of G_x under ψ is precisely $x\,(G \ltimes X)\,x$. The proof is then concluded by invoking Lemma 3.1.3. ☐

3.2. **A Polish topology on quotient spaces.** Suppose in this subsection that G is a Polish groupoid, which is moreover regular. Equivalently the topology of G is (globally) Polish. The following lemma is proved in [42, page 362].

Lemma 3.2.1 (Ramsay)**.** *Suppose that G is a regular Polish groupoid. If U is an open subset of G containing the set of objects G^0, then there is an open subset V of G containing the set of objects G^0 such that $VV \subset U$.*

Fix $x \in G^0$. If V is a neighborhood of G^0 in G, define the set

$$A_{V,x} = \left\{ (\rho, \gamma) \in Gx \times Gx : \rho\gamma^{-1} \in V \right\}.$$

Observe that, if $\gamma \in Gx$, then the collection of open subsets of the form $V\gamma$, where V is an open neighborhood of $r(\gamma)$ in G, is a basis of neighborhoods of γ in Gx. It follows from this observation and Lemma 3.2.1 that the collection

$$\mathcal{U}_x = \left\{ A_{V,x} : V \text{ is a neighborhood of } G^0 \text{ in } G \right\}$$

generates a uniformity compatible with the topology of Gx.

Suppose now that H is a closed subgroup of xGx, and consider the right action of H on Gx by translation. Denote by π the quotient map $Gx \to Gx/H$. Observe that π is continuous and open. If V is a neighborhood of G^0 in G define

$$A_{V,x,H} = \left\{ (\pi(\gamma), \pi(\rho)) \in Gx/H \times Gx/H : \rho h \gamma^{-1} \in V \text{ for some } h \in xGx \right\}.$$

As before the collection

$$\mathcal{U}_{x,H} = \left\{ A_{V,x,H} : V \text{ is a neighborhood of } G^0 \text{ in } G \right\}$$

generates a uniformity compatible with the topology of Gx/H.

Proposition 3.2.2. *The quotient Gx/H is a Polish space.*

Proof. The topology on Gx/H is induced by a countably generated uniformity, and hence it is metrizable. Since the quotient map $\pi : Gx \to Gx/H$ is continuous and open, it follows from [18, Theorem 2.2.9] that Gx/H is Polish. □

Proposition 3.2.3. *Suppose that G is a regular Polish groupoid, and $x \in G^0$. Denote by π the quotient map*

$$\pi : Gx \to Gx/xGx.$$

The following statements are equivalent:

1. *The orbit $[x]$ of x is a G_δ subset of G^0;*
2. *The map $\phi_x : Gx/xGx \to [x]$ defined by $\phi_x(\pi(\gamma)) = r(\gamma)$ is a homeomorphism.*

Proof. The quotient space Gx/xGx is Polish by Proposition 3.2.2. Therefore if ϕ_x is a homeomorphism, then $[x]$ is Polish, and hence a G_δ subset of G^0 by [28, Theorem 3.11]. The converse implication follows from Lemma 3.1.3. □

3.3. G_δ orbits.

Lemma 3.3.1. *Suppose that G is a Polish groupoid, and $(U_n)_{n\in\omega}$ is an enumeration of a basis of nonempty open subsets of G^0. If G has a dense orbit, then every element of $\bigcap_n [U_n]$ has dense orbit.*

The proof of Lemma 3.3.1 is immediate. Recall that $[U_n]$ denotes the G-saturation $r\left[s^{-1}[U_n]\right]$ of U_n.

Lemma 3.3.2. *Suppose that G is a Polish groupoid. Define the equivalence relation $\overline{E}$ on G^0 by $(x, y) \in \overline{E}$ iff the orbits of x and y have the same closure. The equivalence relation $\overline{E}$ is G_δ and contains E_G.*

Proof. Suppose that $(U_n)_{n\in\omega}$ is an enumeration of a countable open basis of G^0. We have that $(x,y)\in\overline{E}$ if and only if $\forall n\in\omega$, $x\in[U_n]$ iff $y\in[U_n]$. It follows that $\overline{E}$ is G_δ. □

Lemma 3.3.3. *Suppose that G is a Polish groupoid such that every orbit of G is G_δ. If $x,y\in G^0$ are such that $[x]\neq[y]$ and $[y]\cap\overline{[x]}\neq\varnothing$ then $\overline{[y]}\cap[x]=\varnothing$. Equivalently the quotient space G^0/E_G is T_0*

Proof. After replacing G with the restriction of G to $\overline{[x]}$ we can assume that $\overline{[y]}\subset\overline{[x]}=X$. Denote by $(U_n)_{n\in\omega}$ an enumeration of a basis of nonempty open subsets of G^0. By Lemma 3.3.1, every element of $\bigcap_n[U_n]$ has dense orbit. Since $[x]\cap[y]=\varnothing$, $[y]$ is not dense in X (otherwise it would be comeager and it would intersect $[x]$). It follows that, for some $n\in\omega$, $y\notin[U_n]$ and hence $[y]\cap U_n=\varnothing$. This shows that $\overline{[y]}\subset X\backslash U_n$. On the other hand U_n is invariant dense open and $[x]$ is comeager, hence $[x]\subset U_n$. This shows that $\overline{[y]}\cap[x]=\varnothing$. □

Lemma 3.3.4. *Suppose that G is a Polish groupoid, and X is a Polish G-space. If every orbit is G_δ, then E_G^X is smooth.*

Proof. By Proposition 2.7.1 we can assume that $X=G^0$ and $G\curvearrowright G^0$ is the standard action. Observe that if $x,y\in G^0$, then $[x]=[y]$ if and only if $[x]$ and $[y]$ have the same closure. This shows that the map $x\mapsto\overline{[x]}$ from G^0 to the space $F(G^0)$ of closed subsets of x endowed with the Effros Borel structure is a reduction from E_G^X to equality in G^0. It remains to observe that such a map is Borel. In fact if U is an open subset of G^0, then

$$\left\{x\in G^0:\overline{[x]}\cap U\neq\varnothing\right\}=[U]=r\left[s^{-1}[U]\right]$$

is open. □

Proposition 3.3.5. *Suppose that G is a Polish groupoid, and X is a Polish G-space. The following statements are equivalent:*

(1) *Every orbit is G_δ;*
(2) *The orbit equivalence relation E_G^X is G_δ;*
(3) *The quotient space X/E_G^X is T_0.*
(4) *The quotient topology generates the quotient Borel structure*

Proof. In view of Proposition 2.7.1 we can assume without loss of generality that $X=G^0$ and $G\curvearrowright G^0$ is the standard action.

(1)⇒(2): Consider the equivalence relation $\overline{E}$ defined as in 3.3.2. Suppose that $x,y\in X$ are such that $(x,y)\in\overline{E}$. It follows that $[x]$ and $[y]$ are both dense subsets of $Y=\overline{[x]}=\overline{[y]}$. Since both the orbit of x and y are G_δ, $[x]$ and $[y]$ are comeager subsets of Y. It follows that they are not disjoint, and hence $[x]=[y]$. This shows that $E_G=\overline{F}$ and in particular E_G is G_δ.

(2)⇒(1): Obvious.

(2)⇒(3): Follows from Lemma 3.3.3.

(3)⇒(1): Since the quotient map $\pi : X \to X/E_G$ is continuous and open, X/E_G has a countable basis $\{U_n : n \in \omega\}$. If $x \in X$ then

$$[x] = \bigcap \left\{ \pi^{-1}[U_n] : n \in \omega,\, \pi(x) \in U_n \right\}.$$

This shows that $[x]$ is G_δ.

(3)⇒(4): The Borel structure generated by the quotient topology is separating and countably generated. By [35, Theorem 4.2] it must coincide with the quotient Borel structure.

(4)⇒(3): Observe that the orbits are Borel. Therefore the quotient Borel structure is separating and hence the quotient topology separates points, i.e. it is T_0.

□

The equivalence of the conditions in Proposition 3.3.5 has been proved in [42, Theorem 2.1] under the additional assumption that the orbit equivalence relation is F_σ.

3.4. **The Glimm-Effros dichotomy.** Denote by E_0 the orbit equivalence relation on 2^ω defined by $(x, y) \in E_0$ iff $x(n) = y(n)$ for all but finitely many $n \in \omega$. Observe that E_0 can be regarded as the (principal) Polish groupoid associated with the free action of $\bigoplus_{n\in\omega} \mathbb{Z}/2\mathbb{Z}$ on $\prod_{n\in\omega} \mathbb{Z}/2\mathbb{Z}$ by translation. The proof of the following result is contained in [42, Section 4]. An exposition of the proof in the case of Polish group actions can be found in [18, Theorem 6.2.1].

Proposition 3.4.1. *Suppose that G is a Polish groupoid. If E_G is dense and meager in $G^0 \times G^0$, then $E_0 \sqsubseteq_c G$.*

Recall that $E_0 \sqsubseteq_c G$ means that there is an injective continuous functor $F : E_0 \to G$ such that the restriction of F to the set of objects is a Borel reduction from E_0 to E_G; see Subsection 2.8. One can then obtain the following consequences:

Proposition 3.4.2. *Suppose that G is a Polish groupoid. If G has no G_δ orbits, then $E_0 \sqsubseteq_c G$.*

Proof. After replacing G with the restriction of G to a class of the equivalence relation $\overline{E}$ defined as in Lemma 3.3.2, we can assume that every orbit is dense. By Theorem 3.1.2 every orbit is meager. It follows from Lemma 2.9.1 that E_G is meager. One can now apply Proposition 3.4.1. □

Theorem 3.4.3. *Suppose that G is a Polish groupoid. If every G_δ orbit is F_σ, then either E_G is G_δ or $E_0 \sqsubseteq_c G$.*

Proof. Suppose that $E_0 \not\sqsubseteq_c G$. In particular for every G_δ subspace Y of G^0, $E_0 \not\sqsubseteq_c G_{|Y}$. Denote by $\overline{E}$ the equivalence relation defined as in Lemma 3.3.2. If $y \in X$ define $Y = [y]_{\overline{E}}$ and observe that Y is an E_G-invariant G_δ subset of G^0. Moreover every E_G-orbit contained in Y is dense in Y. Since E_0 is not continuously reducible to G, by Proposition 3.4.2 there is $z \in Y$ such that $[z]_{E_G}$ is a dense G_δ subset of Y. In particular $[z]_{E_G}$ is G_δ subset of G^0. Therefore by assumption also $Y \setminus [z]_G$ is G_δ. Since every orbit of Y is dense, $Y \setminus [z]_{E_G}$ must be empty and $[z]_{E_G} = Y = [y]_{\overline{E}}$ is G_δ. This shows that every orbit of G is G_δ and hence E_G is G_δ by Proposition 3.3.5. □

Corollary 3.4.4. *Suppose that G and H are Polish groupoids such that every G_δ orbit is F_σ. If E_G and E_H are Borel reducible to E_0, then $G \leq H$ if and only if $E_G \leq E_H$.*

Proof. Suppose that $E_G \leq E_H$. If E_H is smooth then the conclusion follows from Corollary 2.10.10. If E_G is not smooth then $G \sim_B H \sim_B E_0$ by Theorem 3.4.3; see Definition 2.8.3. □

We can combine Proposition 3.3.5 with Theorem 3.4.3 to get the following result. It was obtained in [42] under the additional assumption that the orbit equivalence relation E_G is F_σ. The notion of nonatomic and ergodic Borel measure with respect to an equivalence relation can be found in [18, Definition 6.1.5].

Theorem 3.4.5. *Suppose that G is a Polish groupoid such that every G_δ orbit is F_σ. The following statements are equivalent:*

1. *There is an orbit which is not G_δ;*
2. *$E_0 \sqsubseteq_c G$;*
3. *$E_0 \leq_B E_G$;*
4. *There is an E_G-nonatomic E_G-ergodic Borel probability measure on G^0;*
5. *E_G is not smooth;*
6. *E_G is not G_δ;*
7. *Some orbit is not open in its closure.*

Proof. The implication (1)⇒(2) follows from Theorem 3.4.3. The implication (2)⇒(3) is obvious. For (3)⇒(4), observe that if $f : 2^\omega \to X$ is a Borel reduction from E_0 to E_G, μ is the product measure on 2^ω, and ν is the push-forward of μ under f, then ν is an E_G-nonatomic and E_G-ergodic Borel probability measure on G^0. The implication (4)⇒(5) follows from [18, Proposition 6.1.6]. By Lemma 3.3.4 (5) implies 6). The implication (6)⇒(1) is contained in Proposition 3.3.5. Since a set that is open in its closure is G_δ, the implication (1)⇒(7) is obvious. Let us show that (7)⇒(1). Suppose that every orbit is G_δ, and fix $x \in G^0$. After replacing G with its restriction to the closure of the orbit of x, we can assume that x has dense orbit. Therefore $[x]$ is a dense G_δ in x. Since $[x]$ is by assumption also F_σ, $[x] = \bigcup_n F_n$ where the F_n's are closed in X. Being $[x]$ nonmeager in X, there is an open subset U of X contained in F_n for some $n \in \omega$. Hence $[x] = [U]$ is open. □

4. Better topologies

4.1. **Polishability of Borel G-spaces.**

Theorem 4.1.1. *Suppose that G is a Polish groupoid. Every Borel G-space is Borel G-isomorphic to a Polish G-space. Equivalently if X is a Borel G-space, then there is a Polish topology compatible with the Borel structure of G that makes the action of G on X continuous.*

Theorem 4.1.1 answers a question of Ramsay from [43]. The rest of this subsection is dedicated to the proof of Theorem 4.1.1. The analogous statement for actions of Polish groups is proved in a similar way in [3, Theorem 5.2.1]. Fix a countable basis $\mathcal{A}$ of Polish open subsets of G. Suppose that X is a Polish G-space. We want to define a topology t on X such that

(1) t is Polish,
(2) the action $G \curvearrowright (X, t)$ is continuous, i.e. the anchor map $p : X \to G^0$ is continuous and
$$\begin{array}{ccc} G \ltimes X & \to & X \\ (\gamma, x) & \mapsto & \gamma x \end{array}$$
is continuous,
(3) t generates the Borel structure of X.

By Lemma 2.10.5 and [3, 5.1.3 and 5.1.4] there exists a countable Boolean algebra $\mathcal{B}$ of Borel subsets of X satisfying the following conditions:

- For all $B \in \mathcal{B}$ and $U \in \mathcal{A}$, $B^{\triangle U} \in \mathcal{B}$;
- The topology t' generated by the basis $\mathcal{B}$ is Polish.

Observe that the identity function from X with its original Borel structure to (X, t') is Borel measurable, and hence a Borel isomorphism by [28, Theorem 15.1]. It follows that t' generates the Borel structure of X. Define $\mathcal{S}$ to be the set
$$\left\{ B^{\triangle U} : B \in \mathcal{B}, U \in \mathcal{A} \right\},$$
and t to be the topology on X having $\mathcal{S}$ as subbasis.

Claim. The action $G \curvearrowright (X, t)$ is continuous.

Proof. If $V \in \mathcal{A}$ then
$$p^{-1}[s[V]] = X^{\triangle V} \in \mathcal{S}.$$
This shows that $p : X \to G^0$ is t-continuous. Let us now show that the map $G \ltimes X \to X$ is t-continuous. Suppose that $B \in \mathcal{B}$, $U \in \mathcal{A}$, and $(\gamma_0, x_0) \in G \ltimes X$ is such that $\gamma_0 x_0 \in B^{\triangle U}$. By Lemma 2.10.3 there are $W, V \in \mathcal{A}$ such that $VW^{-1} \subset U$, $x_0 \in B^{\triangle V}$, and $\gamma_0 \in W$. We claim that $\gamma x \in B^{\triangle U}$ for every $x \in B^{\triangle V}$ and $\gamma \in W$. Fix $x \in B^{\triangle V}$ and $\gamma \in W$ and observe that $V\gamma^{-1} \subset VW^{-1} \subset U$ and hence it follows from Lemma 2.10.3 that $\gamma x \in B^{\triangle U}$. This shows that the action is continuous. ☐

Claim. The space (X, t) is T_1.

Proof. Pick distinct points x, y of X. If $p(x) \neq p(y)$ then there are disjoint $V, W \in \mathcal{A}$ such that $p(x) \in V$ and $p(y) \in W$. Thus $p^{-1}[V]$ and $p^{-1}[W]$

are open sets separating x and y. Suppose that $p(x)=p(y)$. Consider the function $f:Gp(x)\to X\times X$ defined by

$$f(\gamma)=(\gamma x,\gamma y)\,.$$

Observe that f is Borel when $X\times X$ is endowed with the $t'\times t'$ topology. By [28, Theorem 8.38] there is a dense G_δ subset Q of Gx such that the restriction of f to Q is $(t'\times t')$-continuous. Let $\gamma_0\in Q$. Since $\mathcal{B}$ is a basis for the Polish topology t' on X there are disjoint elements B,C of $\mathcal{B}$ such that $\gamma_0x\in B$ and $\gamma_0y\in C$. Since f is $(t'\times t')$-continuous on Q there is $U\in\mathcal{A}$ such that $Up(x)\neq\varnothing$ and

$$f\left[Up(x)\cap Q\right]\subset B\times C.$$

Thus $\forall^*\gamma\in Up(x)$, $\gamma x\in B$ and $\gamma y\in C$. This shows that

$$x\in B^{\triangle U},\ y\in C^{\triangle U},\ y\notin B^{\triangle U},\text{ and } x\notin C^{\triangle U}.$$

This concludes the proof that (X,t) is T_1. □

Claim. The space (X,t) is regular.

Proof. Suppose that $B\in\mathcal{B}$ and $U\in\mathcal{A}$. Pick $x_0\in B^{\triangle U}$. It is enough to show that there is a t-open subset N of $B^{\triangle U}$ containing x_0 such that the t-closure of N is contained in $B^{\triangle V}$. Since $x_0\in B^{\triangle U}$ by Lemma 2.10.3 there are $W_1,V_1\in\mathcal{A}$ such that $V_1W_1^{-1}\cup V_1\subset U$, $p(x_0)\in W_1$, and $x_0\in B^{\triangle V_1}$. Since $x_0\in B^{\triangle V_1}$ again by Lemma 2.10.3 there are $V_2,W_2\in\mathcal{A}$ such that $V_2W_2^{-1}\subset V_1$, $p(x_0)\in W_2$, and $x_0\in B^{\triangle V_2}$. Define $W\in\mathcal{A}$ such that

$$p(x_0)\in W\subset W_1^{-1}\cap W_2.$$

Consider

$$N=B^{\triangle V_2}\cap p^{-1}s\left[W\right]$$

and observe that N is a t-open subset of X containing x_0. We claim that the closure of N is contained in $B^{\triangle U}$. Define $F=\left(B^{\triangle V_2}\right)^{*W}$ and observe that F is relatively closed in $p^{-1}s\left[W\right]$ by Lemma 2.10.2(4). We claim that

$$N\subset F\subset B^{\triangle U}.$$

Suppose that $x\in N$. If $\gamma\in Wp(x)$ we have that

$$V_2\gamma^{-1}\subset V_2W^{-1}\subset V_2W_2^{-1}\subset V_1.$$

Therefore $\gamma x\in B^{\triangle V_1}$. Being this true for every $\gamma\in Wp(x)$, $x\in\left(B^{\triangle V_1}\right)^{*W}=F$. Suppose now that $x\in F$ and pick $\gamma\in Wp(x)$ such that $\gamma x\in B^{\triangle V_1}$. We thus have

$$V_1\gamma\subset V_1W\subset V_1W_1^{-1}\subset U$$

which implies by Lemma 2.10.3 that $x=\gamma^{-1}(\gamma x)\in B^{\triangle U}$. This concludes the proof that $N\subset F\subset B^{\triangle U}$. We will now show that the t-closure of N is contained in $B^{\triangle U}$. It is enough to show that if $x\notin B^{\triangle U}$ then there is a t-open neighborhood of x disjoint from N. This is clear if $p(x) \notin s[W]$. Suppose now that $p(x) \in s[W]$. Since F is relatively closed in $p^{-1}s[W]$ and

$$N \subset F \subset B^{\triangle V} \cap p^{-1}s[W]$$

we have that $p^{-1}s[W] \setminus F$ is an open subset of X containing x and disjoint from N. This concludes the proof that the closure of N is contained in $B^{\triangle V}$. We have thus found an open neighborhood N of x whose closure is contained in $B^{\triangle V}$. This concludes the proof that (X, t) is regular. □

Claim. The space (X, t) is strong Choquet.

Proof. Define $\mathcal{C}$ to be the (countable) set of nonempty finite intersections of elements of $\mathcal{S}$ and observe that $\mathcal{C}$ is a basis for (X, t). Fix a well ordering E of the countable set $\mathcal{C} \times \mathcal{B} \times \mathcal{A}$. Let d' be a complete metric on X compatible with the Polish topology t'. We want to define a strategy for Player II in the strong Choquet game; see [28, Section 8.D]. Suppose that Player I plays t-open sets N_i for $i \in \omega$ and $x_i \in N_i$. At the i-th turn Player II will choose an element (M_i, B_i, U_i) of $\mathcal{C} \times \mathcal{B} \times \mathcal{A}$ in such a way that the following properties hold:

(1) $x_i \in M_i$;
(2) The t-closure of M_i is contained in N_i;
(3) The closure of U_{i+1} in U_0 is contained in U_i;
(4) The t'-closure of B_{i+1} is contained in B_i;
(5) The d'-diameter of B_i is less than 2^{-i};
(6) The d_{U_0}-diameter of U_i is less than 2^{-i} for $i \geq 1$, where d_{U_0} is a compatible complete metric on U_0;
(7) $M_i \subset B_i^{\triangle U_i}$.

Player II strategy is the following: At the i-th turn pick the E-least tuple (M_i, B_i, U_i) in $\mathcal{C} \times \mathcal{B} \times \mathcal{A}$ satisfying properties (1)–(7). We need to show that the set of such tuples is nonempty. Observe that $x_i \in N_i \subset M_{i-1} \subset B_{i-1}^{\triangle U_{i-1}}$. Thus $U_{i-1}p(x_i) \neq \varnothing$ and $\exists^* \gamma \in U_{i-1}p(x_i)$ such that $\gamma x_i \in B_{i-1}$. Since $\mathcal{B}$ is a basis for (X, t') and $\mathcal{A}$ is a basis for G we can find B_i and U_i such that

- (3)–(6) hold.
- $U_i p(x_i) \neq \varnothing$, and
- $\exists^* \gamma \in U_i p(x_i)$ such that $\gamma x_i \in B_i$.

Consider $M = B_i^{\triangle U_i} \cap N_i$ and observe that M is a t-open set containing x_i. Since (X, t) is regular there is $M_i \in \mathcal{C}$ such that $x_i \in M_i$ and the closure of M_i is contained in $N_i \cap B_i^{\triangle U_i}$. This ensures that (1),(2),(7) are satisfied. We now show that this gives a winning strategy for Player II. For every $i \in \omega$ we have that $x_i \in M_i \subset B_i^{\triangle U_i}$ and hence there is $\gamma_i \in U_i p(x_i)$ such that $\gamma_i x_i = y_i \in B_i$. Define γ to be the limit of the sequence $(\gamma_i)_{i \in \omega}$ in U_0 and y to be the t'-limit of the sequence $(y_i)_{i \in \omega}$ in Y. Observe that

$$p(y) = \lim_i p(y_i) = \lim_i r(\gamma_i) = r(\gamma).$$

Define $x = \gamma^{-1}y \in X$ and observe that x is the t-limit of the sequence $(x_i)_{i\in\omega}$. Fix $i \in \omega$. For $j > i$ we have that $x_j \in N_j \subset M_i$ and hence x is contained in the t-closure of M_i, which is in turn contained in N_i. This shows that $x \in \bigcap_{i\in\omega} N_i$, concluding the proof that Player II has a winning strategy in the strong Choquet game in (X, t). □

The proof of Theorem 4.1.1 is finished recalling that a regular T_1 strong Choquet space is Polish [28, Theorem 8.18].

4.2. **Finer topologies for Polish G-spaces.**

Theorem 4.2.1. *Suppose that G is a Polish groupoid, and (X, τ) is a Polish G-space. Assume that $V \subset G$ is an open Polish subset, $P \subset X$ is $\mathbf{\Sigma}^0_\alpha$ for some $\alpha \in \omega_1$, and $Q = P^{\triangle V}$.There is a topology t on X such that:*

1. *t is a Polish topology;*
2. *t is finer that τ;*
3. *Q is t-open,*
4. *The action of G on (X, t) is continuous;*
5. *t has a countable basis $\mathcal{B}$ such that for every $B \in \mathcal{B}$ there is $n \in \omega$ such that B is $\mathbf{\Sigma}^0_{\alpha+n}$ with respect to τ.*

The analogous statement for actions of Polish groups is proved in a similar way in [3, Theorem 5.1.8]. Let $\mathcal{A}$ be a countable basis of Polish open subsets for G containing V and let $\mathcal{D}$ be a countable basis for (X, τ). By Lemma 2.10.6 and [3, 5.1.3, 5.1.4] there is a countable Boolean algebra $\mathcal{B}$ of subsets of X satisfying the following:

1. For $B \in \mathcal{B}$ and $U \in \mathcal{A}$, $B^{\triangle V} \in \mathcal{A}$;
2. $P \in \mathcal{B}$;
3. $\mathcal{D} \subset \mathcal{B}$;
4. $\mathcal{B}$ is a basis for a Polish topology t';
5. For every $B \in \mathcal{B}$, there is $n \in \omega$ such that B is $\mathbf{\Sigma}_{\alpha+n}$ with respect to τ.

Define

$$\mathcal{S} = \left\{ B^{\triangle V} : B \in \mathcal{B}, V \in \mathcal{A} \right\},$$

and

$$\mathcal{S}^* = \mathcal{S} \cup \mathcal{D}.$$

Consider the topology t on X having $\mathcal{S}^*$ as a subbasis. We claim that t is a Polish topology finer that τ and coarser that t' making the action continuous. Clearly t is finer that τ and in particular $p : (X, t) \to G^0$ is continuous. The proof that the action is continuous and that t is a Polish topology is analogous to the proof of Theorem 4.1.1. The following corollary can be obtained from Theorem 4.2.1 together with [3, Subsection 5.1.3].

Corollary 4.2.2. *Suppose that G is a Polish groupoid, and (X, τ) is a Polish G-space. If $\mathcal{J}$ is a countable collection of G-invariant Borel subsets of X, then there is a Polish topology t on X finer than τ and making the action continuous such that all elements of $\mathcal{J}$ are t-clopen.*

5. Borel orbit equivalence relations

5.1. **A Borel selector for cosets.** Suppose that G is an open Polish groupoid. Denote by $F(G)$ the (standard) Borel space of closed subsets of G endowed with the Effros Borel structure; see Appendix 9. A similar proof as [28, Theorem 12.13] shows that there is a Borel function

$$\sigma : F(G)\backslash\{\varnothing\} \to G$$

such that $\sigma(A) \in A$ for every nonempty closed subset A of G. Denote by $S(G)$ the Borel space of closed subgroupoids of G. This is the Borel subset of $F(G)$ containing the closed subsets H of G such that for $\gamma, \rho \in H$, $\gamma^{-1} \in H$ and if $r(\gamma) = s(\rho)$ then $\rho\gamma \in H$. If $H \in S(G)$ denote by $\sim_H$ the equivalence relation on G defined by $\gamma \sim_H \rho$ iff $\gamma = \rho h$ for some $h \in H$ or, equivalently, $\gamma H = \rho H$.

Proposition 5.1.1. *The relation $\sim$ on $G\times S(G)$ defined by $(\gamma, H) \sim (\gamma', H')$ iff $H = H'$ and $\gamma H = \gamma' H'$ has a Borel transversal T.*

Proof. Define the map f from $S(G) \times G$ to $F(G)$ by $f(\gamma, H) = \gamma H$. We claim that f is Borel. Let us show that if U is an open subset of G then the set

$$A_U = \{(\gamma, H) \in G \times S(G) : \gamma H \cap U \neq \varnothing\}$$

is Borel. Since the set

$$\left\{(\rho, \gamma, H) \in G \times G \times S(G) : \gamma^{-1}\rho \in H \text{ and } \rho \in U\right\}$$

is Borel, its projection A_U on the last two coordinates is analytic. We want to show that A_U is co-analytic. Fix a countable basis of Polish open sets $\{U_n : n \in \omega\}$ for G. Observe that $(H, \gamma) \in A_U$ if and only if there is $n \in \omega$ such that $\gamma U_n \subset U$ and $U_n \cap H \neq \varnothing$. It is now enough to show that $\{\gamma \in G : \gamma U_n \subset U\}$ is co-analytic. This follows from the fact that

$$\{(\gamma, \rho) \in G \times U_n : \text{either } r(\rho) \neq s(\gamma) \text{ or } r(\rho) = s(\gamma) \text{ and } \gamma\rho \in U\}$$

is a Borel set and it co-projection on the first coordinate is $\{\gamma \in G : U_n\gamma \subset U\}$. If now $\sigma : F(G)\backslash\{\varnothing\} \to G$ is a Borel map such that $\sigma(A) \in A$ for every nonempty closed subset A of G, define $g(\gamma, H) = ((\sigma \circ f)(\gamma, H), H)$. Observe that g is a Borel selector for $\sim$. Therefore the set

$$T = \{(\gamma, H) : g(\gamma, H) = (\gamma, H)\}$$

is a Borel transversal for $\sim$. $\square$

Corollary 5.1.2. *If G is a Polish groupoid, and X is a Borel G-space, then the orbits are Borel subsets of X.*

Proof. Observe that the stabilizer G_x is a closed subgroup of $p(x)Gp(x)$ by [28, Theorem 9.17]. Consider a Borel transversal T_x for the equivalence relation $\sim_{G_x}$. The function $\gamma \mapsto \gamma x$ from $T \cap Gx$ to X is a 1:1 Borel function from T_x onto the orbit of x. This shows that the orbit of x is Borel by [28, Theorem 15.1]. $\square$

5.2. **Borel orbit equivalence relations.** Suppose that G is a Polish groupoid, and X is a Polish G-space. If $x \in X$ then Lemma 2.10.5 and [3, 5.1.3 and 5.1.4] show that there is a sequence $(B_{x,n})_{n\in\omega}$ of Borel subsets of X such that $[x] = B_{x,0}$ and
$$\mathcal{B}(x) = \{B_{x,n} : n \in \omega\}$$
is a Boolean algebra that is a basis for a topology $t(x)$ on X making the action continuous, and such that $B^{\triangle U} \in \mathcal{B}(x)$ whenever $B \in \mathcal{B}(x)$ and $U \in \mathcal{A}$. It is implicit in the proof of Lemma 2.10.5 and [3, 5.1.3, 5.1.4] that, under the additional assumption that the orbit equivalence relation E_G^X is Borel, the dependence of the sequence $(\mathcal{B}_{x,n})_{n\in\omega}$ from x is Borel, i.e. the relation
$$\mathcal{B}(y,x,n) \Leftrightarrow y \in \mathcal{B}_{x,n}$$
is Borel. This concludes the proof of the following lemma; see also [3, Lemma 7.1.3].

Lemma 5.2.1. *Suppose that G is a Polish groupoid, and X is a Polish G-space. Assume that $\mathcal{A}$ is a countable basis of Polish open subsets of G. If the orbit equivalence relation E_G^X is Borel, then there is a Borel subset $\mathcal{B}$ of $X \times X \times \omega$ such that, letting*
$$B_{x,n} = \{y : (y,x,n) \in \mathcal{B}\}$$
and
$$\mathcal{B}(x) = \{B_{x,n} : n \in \omega\},$$
for every $x \in X$ the following hold:

1. *$[x] = B_{x,0}$;*
2. *$B^{\triangle U} \in \mathcal{B}(x)$ for every $B \in \mathcal{B}(x)$, and $U \in \mathcal{A}$;*
3. *$\mathcal{B}(x)$ is a Boolean algebra;*
4. *$\mathcal{B}(x)$ is a basis for a Polish topology $t(x)$ on X making X a Polish G-space.*

The following result provides a characterization of the Borel G-spaces with Borel orbit equivalence relation. The analogous result for Polish group actions is [3, Theorem 7.1.2].

Theorem 5.2.2. *Suppose that G is a Polish groupoid, and X is a Borel G-space. The following statements are equivalent*

1. *The function*
$$\begin{array}{rcl} X & \to & F(G) \\ x & \mapsto & G_x \end{array}$$
is Borel;
2. *The function*
$$\begin{array}{rcl} X \times X & \to & F(G) \\ (x,y) & \mapsto & G_{x,y} \end{array}$$
is Borel;

(3) *The orbit equivalence relation E_G^X is Borel.*

Recall that, for $x, y \in G^0$, G_x denotes the (closed) stabilizer

$$\{\gamma \in Gp(x) : \gamma x = x\}$$

and while $G_{x,y}$ is the set

$$\{\gamma \in Gp(x) : \gamma x = y\} ;$$

see Subsection 2.5.

Proof. By Theorem 4.1.1 we can assume without loss of generality that X is a Polish G-space. Fix a countable basis $\mathcal{A} = \{U_n : n \in \omega\}$ of nonempty Polish open subsets of G. Denote also by $T \subset G \times S(G)$ a Borel transversal for the relation $(\gamma, H) \sim (\gamma', H')$ iff $H = H'$ and $\gamma H = \gamma H'$ as in Proposition 5.1.1.

(1)⇒(2): Fix a nonempty open subset U of G. It is enough to show that the set

$$\{(x, y) \in X \times X : U \cap G_{x,y} \neq \varnothing\}$$

is co-analytic. Observe that $G_{x,y} \cap U \neq \varnothing$ if and only if there is a unique $\gamma \in G$ such that $s(\gamma) = p(x)$, $(\gamma, G_x) \in T$, and $\gamma G_x \cap U \neq \varnothing$. Moreover $\gamma G_x \cap U \neq \varnothing$ if and only if there is $n \in \omega$ such that $\gamma U_n \subset U$ and $U_n \cap G_x \neq \varnothing$. Fix $n \in \omega$ and recall that by the proof of Proposition 5.1.1 $\{\gamma \in G : \gamma U_n \subset U\}$ is co-analytic. This concludes the proof that

$$\{(x, y) \in X \times X : U \cap G_{x,y} \neq \varnothing\}$$

is co-analytic.

(2)⇒(1): Obvious.

(1)⇒(3): Observe that $(x, y) \in E_G$ if and only if there is a unique $\gamma \in T$ such that $(\gamma, G_x) \in T$ and $r(\gamma) = y$.

(3)⇒(1): Suppose that $\mathcal{B}$, $\mathcal{B}(x)$, $t(x)$, and $B_{x,n}$ for $x \in X$ and $n \in \omega$ are defined as in Lemma 5.2.1. Observe that the orbit $[x] = B_{x,0}$ is open in $t(x)$. It follows from Lemma 3.1.3 that the map

$$\begin{aligned} Gp(x)/G_x &\to [x] \\ \gamma G_x &\mapsto \gamma x \end{aligned}$$

is a $t(x)$-homeomorphism. We want to show that for every $U \in \mathcal{A}$ the set

$$\{x \in X : G_x \cap U \neq \varnothing\}$$

is Borel. It is enough to show that for every basic nonempty U, V the set

$$\{x \in X : UG_x \cap V \neq \varnothing\}$$

is co-analytic. We claim that $UG_x \cap V \neq \varnothing$ iff $\exists U_m \subset U$ such that $\forall B \in \mathcal{B}(x)$, $x \in B^{\triangle U_m}$ implies $x \in B^{\triangle V}$. In fact suppose that $UG_x \cap V \neq \varnothing$ and pick m such that $U_m \subset U$ and $U_m\gamma \subset V$ for some

$\gamma \in G_x$. If $x \in B^{\triangle U_m}$ then $x = \gamma^{-1}x \in B^{\triangle U_m \gamma}$ and hence $x \in B^{\triangle V}$ by Lemma 2.10.3. Conversely suppose that $UG_x \cap V = \varnothing$ and hence

$$\{\gamma x : \gamma \in U\} \cap \{\gamma x : \gamma \in V\} = \varnothing.$$

Fix $m \in \omega$. Since the map

$$\begin{array}{rcl} Gp(x)\,/G_x & \to & [x] \\ \gamma G_x & \mapsto & \gamma x \end{array}$$

is a $t(x)$-homeomorphism, the set $\{\gamma x : \gamma \in U_m\}$ is open in $[x]$. Thus there is $B \in \mathcal{B}(x)$ such that

$$x \in B \subset \{\gamma x : \gamma \in U_m\}.$$

Moreover

$$\{\gamma \in Gp(x) : \gamma x \in B\}$$

is an open subset of $Gp(x)$. Therefore there is $k \in \omega$ such that $U_k \subset U_m$ and

$$U_k p(x) \subset \{\gamma \in Gp(x) : \gamma x \in B\}.$$

In particular $x \in B^{\triangle U_k}$ but $x \notin B^{\triangle V}$.

□

6. Universal actions

Suppose that G is a Polish groupoid. The space G is fibred over the space of objects G^0 via the source map $r : G \to G^0$. One can then consider the corresponding Effros fibred space $F(G, G^0)$ of closed subsets of G contained in Gx for some $x \in G^0$; see Subsection 2.3. Recall that $F(G, G^0)$ is a standard Borel space fibred over G^0 via the Borel map assigning x to a closed nonempty subset F of xG. Moreover $F(G, G^0)$ has naturally the structure of Borel G-space given by the map

$$(\gamma, F) \mapsto \gamma F$$

for $F \subset s(\gamma)G$, where

$$\gamma F = \{\gamma\rho : \rho \in F\}.$$

Similarly the fibred product

$$\mathop{\ast}_{n\in\omega} F(G, G^0) = \left\{(F_n)_{n\in\omega} \in F(G, G)^\omega : \exists x \in G^0\ \forall n \in \omega,\ F_n \subset xG\right\}$$

is naturally a Borel G-space with respect to the coordinate-wise action of G

$$\gamma(F_n)_{n\in\omega} = (\gamma F_n)_{n\in\omega}.$$

We want to show that the Borel G-space $\ast_{n\in\omega} F(G, G^0)$ is a universal Borel G-space. This means that if X is any Borel G-space, then there is a Borel G-embedding $\varphi : X \to \ast_{n\in\omega} F(G, G^0)$; see Subsection 2.5.

The following lemma is well known. A proof is included for convenience of the reader.

Lemma 6.0.3. *If X is a Polish space, $A \subset X$, and $E(A)$ is the set of $x \in X$ such that for every neighborhood V of x, $V \cap A$ is not meager, then $E(A)$ is closed in X. Moreover A has the Baire property iff $A \bigtriangleup E(A)$ is meager.*

Proof. Clearly $E(A)$ is closed, and if $A \bigtriangleup E(A)$ is meager then A has the Baire property. Observe that if $A, B \subset X$ are such that $A \bigtriangleup B$ is meager, then $E(A) = E(B)$. If A has the Baire property, there is an open subset U of X such that $A \bigtriangleup U$ is meager. Thus $E(A) = E(U)$ is equal to the closure $\overline{U}$ of U. It follows that

$$A \bigtriangleup E(A) \subset (A \bigtriangleup U) \cup \left(\overline{U} \backslash U\right)$$

is meager. □

Suppose that X is a Borel G-space. In view of Theorem 4.1.1 we can assume without loss of generality that X is in fact a Polish G-space. Fix a countable open basis $\mathcal{B} = \{B_n : n \in \omega\}$ of nonempty open subsets of X. Assume further that $\mathcal{A}$ is a countable basis of Polish open subsets of G. Define for $n \in \omega$ the fibred Borel map $\varphi_n : X \to F(G, G^0)$ by setting

$$\varphi_n(x) = \left(E\left(\{\gamma \in Gp(x) : \gamma x \in B_n\}\right)\right)^{-1}.$$

Define the Borel fibred map $\varphi : X \to \mathop{\ast}_{n\in\omega} F(G, G^0)$ by $\varphi(x) = (\varphi_n(x))_{n\in\omega}$.

Claim. φ is Borel measurable

It is enough to show that φ_n is Borel measurable for every $n \in \omega$. Suppose that $V \in \mathcal{A}$. We want to show that the set of $x \in X$ such that

$$E\left(\{\gamma \in Gp(x) : \gamma x \in B_n\}\right) \cap V \neq \varnothing$$

is Borel. Observe that

$$E\left(\{\gamma \in Gp(x) : \gamma x \in B_n\}\right) \cap V \neq \varnothing$$

if and only if $\exists W \in \mathcal{A}$ such that $W \subset V$ and

$$\{\gamma \in Gp(x) : \gamma x \in B_n\}$$

is comeager in $Wp(x)$. The set of such elements x of X is Borel by Lemma 2.10.4.

Claim. φ is G-equivariant, i.e. $\varphi(\gamma x) = \gamma\varphi(x)$ for $(\gamma, x) \in G \ltimes X$

It is enough to show that $\varphi_n(\gamma x) = \gamma\varphi_n(x)$ for $n \in \omega$. Observe that

$$\varphi_n(\gamma x) = \left(E\left(\{\rho \in Gr(\gamma) : \rho\gamma x \in B_n\}\right)\right)^{-1}$$

and

$$\varphi_n(x) = \left(E\left(\{\tau \in Gp(x) : \tau x \in B_n\}\right)\right)^{-1}$$

We thus have to prove that

$$\left(E\left(\{\rho \in Gr(\gamma) : \rho\gamma x \in B_n\}\right)\right)^{-1} = \gamma\left(E\left(\{\tau \in Gp(x) : \tau x \in B_n\}\right)\right)^{-1}$$

or equivalently

$$E\left(\{\rho \in Gr(\gamma) : \rho\gamma x \in B_n\}\right) = E\left(\{\tau \in Gp(x) : \tau x \in B_n\}\right)\gamma^{-1}$$

Since $\tau \mapsto \tau\gamma^{-1}$ is a homeomorphism from $Gp(x)$ to $Gr(\gamma)$ we have that

$$\begin{aligned} E\left(\{\tau \in Gp(x) : \tau x \in B_n\}\right)\gamma^{-1} &= E\left(\{\tau \in Gp(x) : \tau x \in B_n\}\gamma^{-1}\right) \\ &= E\left(\{\rho \in Gr(\gamma) : \rho\gamma x \in B_n\}\right) \end{aligned}$$

Claim. φ is injective

Assume that $x, y \in X$ are such that $\varphi(x) = \varphi(y)$. Thus $p(x) = p(y)$ and for every $n \in \omega$

$$\{\gamma \in Gp(x) : \gamma x \in B_n\} \bigtriangleup \{\gamma \in Gp(y) : \gamma y \in B_n\}$$

is meager. Thus $\forall^*\gamma \in Gp(x)$, $\forall n \in \omega$, $\gamma x \in B_n$ iff $\gamma y \in B_n$. Thus for some $\gamma \in Gp(x)$, $\gamma x \in B_n$ iff $\gamma y \in B_n$ for all $n \in \omega$. This implies that $\gamma x = \gamma y$ and hence $x = y$.

7. Countable Borel groupoids

7.1. **Actions of inverse semigroups on Polish spaces.** An *inverse semigroup* is a semigroup T such that every $t \in T$ has a semigroup-theoretic inverse $t^* \in T$. This means that t^* is the unique element of T such that

$$tt^*t = t \text{ and } t^*tt^* = t^*.$$

If T is an inverse semigroup, then the set $E(T)$ of idempotent elements is a commutative subsemigroup of T, and hence a semilattice; see [40, Proposition 2.1.1]. In particular $E(T)$ has a natural order defined by

$$e \leq f \text{ iff } ef = fe = e.$$

Observe that for every $t \in T$ the elements tt^* and t^*t are idempotent.

Suppose that X is a Polish space. The semigroup $\mathcal{H}(X)$ of partial homeomorphisms between open subsets of X is clearly an inverse semigroup.

Definition 7.1.1. An action $\theta : T \curvearrowright X$ of a countable inverse semigroup T on the Polish space X is a semigroup homomorphism $\theta : t \mapsto \theta_t$ from T to $\mathcal{H}(X)$.

Observe that a semigroup homomorphism between inverse semigroups automatically preserves inverses; see [40, Proposition 2.1.1].

7.2. **Étale Polish groupoids.** Suppose that G is a Polish groupoid. A subset u of G is a bisection if the source and range maps restricted to u are injective. A bisection of G is open if it is an open subset of G. It is not difficult to verify that the following conditions are equivalent:

(1) The source and range maps of G are local homeomorphisms from G to G^0;
(2) Composition of arrows in G is a local homeomorphism from G^2 to G;
(3) G has a countable basis of open bisections;
(4) G has a countable inverse semigroup of open bisections that is basis for the topology of G;

(5) G^0 is an open subset of G.

When these equivalent conditions are satisfied, G is called étale Polish groupoid. If G is an étale Polish groupoid, then in particular for every $x \in G^0$ the fiber Gx is a countable discrete subset of G.

7.3. **The groupoid of germs.** Suppose that $\theta : T \curvearrowright X$ is an action of a countable inverse semigroup on a Polish space. We want to associate to such an action an étale Polish groupoid $\mathcal{G}(\theta, T, X)$ that contains all the information about the action. This construction can be found in [10] in the case when X is locally compact.

If $e \in E(T)$ denote by D_e the domain of θ_e. Observe that the domain of θ_t is D_{t^*t} and the range of θ_t is D_{tt^*}. Define Ω to be the subset of $T \times X$ of pairs (u, x) such that $x \in D_{u^*u}$. Consider the equivalence relation $\sim$ on Ω defined by $(u, x) \sim (v, y)$ iff $x = y$ and for some $e \in E(S)$, $ue = ve$ and $x \in D_e$. The equivalence class $[u, x]$ of (u, x) is called the *germ* of u at x. Observe that if e witnesses that $(u, x) \sim (v, x)$ then, after replacing e with u^*uv^*ve we can assume that $e \le u^*u$ and $e \le v^*v$. It can be verified as in [10, Proposition 4.7] that if (u, x) and (v, y) are in Ω and $x = \theta_v(y)$ then $(uv, y) \in \Omega$. Moreover the germ $[uv, y]$ of uv at y depends only on $[x, s]$ and $[y, t]$.

One can then define the groupoid $\mathcal{G}(\theta, T, X) = \Omega/\!\sim$ of germs of the action $S \curvearrowright X$ obtained by setting

- $\mathcal{G}(\theta, T, X)^2 = \{([u, x], [v, y]) : \theta_v(y) = x\}$,
- $[u, x]\,[v, y] = [uv, y]$, and
- $[u, x]^{-1} = [\theta_{u^*}, \theta_u(x)]$.

Observe that the map $x \mapsto [e, x]$ from X to G, where e is any element of $E(S)$ such that $x \in D_e$, is a well-defined bijection from X to the set of objects G^0 of G. Identifying X with G^0 we have that the source and range maps s and r are defined by

$$s\,[u, x] = x$$

and

$$r\,[u, x] = \theta_u(x).$$

We now define the topology of $\mathcal{G}(\theta, T, X)$. For $u \in T$ and $U \subset D_{u^*u}$ open define

$$\Theta\,(u, U) = \{[u, x] \in G : x \in U\}$$

It can be verified as in [10, Proposition 4.14, Proposition 4.15, Corollary 4.16, Proposition 4.17, and Proposition 4.18] that the following hold:

(1) $\mathcal{G}(\theta, T, X)$ is an étale Polish groupoid;
(2) the map $x \mapsto [e, x]$ where e is any element of $E(S)$ such that $x \in D_s$, is a homeomorphism from X onto the space of objects of $\mathcal{G}\,(\theta, T, x)$;
(3) if $u \in T$ and $U \subset D_{u^*u}$ then $\Theta\,(u, U)$ is an open bisection of U, and the map $x \mapsto [u, x]$ is a homeomorphism from U onto $\theta\,(u, U)$;

(4) if $\mathcal{A}$ is a basis for the topology of X, then the collection
$$\{\theta\left(u, A\cap D_{u^*u}\right) : u\in S, A\in \mathcal{A}\}$$
is a basis of open bisections for $\mathcal{G}(\theta, T, X)$.

7.4. **Regularity of the groupoid of germs.** The groupoid of germs $\mathcal{G}(\theta, T, X)$ for an action $\theta : T \curvearrowright X$ is in general not Hausdorff, even when X is locally compact. Here we isolate a condition that ensures that $\mathcal{G}(\theta, T, X)$ is regular.

Define the order $\leq$ on T by setting $u \leq v$ iff $u = vu^*u$. Observe that this extends the order of $E(T)$. Moreover if $u \leq v$ then
$$u^*u = v^*vu^*u^*v^*v = v^*vu^*u$$
and hence $u^*u \leq v^*v$. We say that T is a *semilattice* if it is a semilattice with respect to the order $\leq$ just defined, i.e. for every pair u, v of elements of T there is a largest element $u \wedge v$ below both u and v.

Proposition 7.4.1. *Suppose that T is a semilattice. If there is a subset C of T such that:*

(1) *for every $u \in T$ and $x \in D_{u^*u}$ there is $c \in C$ such that $x \in D_{(u\wedge c)^*(u\wedge c)}$, and*
(2) *for every distinct $c, d \in C$, $\Theta\left(c, D_{c^*c}\right) \cap \Theta\left(d, D_{d^*d}\right) = \varnothing$,*

then the groupoid of germs $\mathcal{G}(\theta, T, X)$ is regular.

Proof. Suppose that $[u, x]$ is an element of $\mathcal{G}(\theta, T, X)$, and W is an open neighborhood of $[u, x]$ in $\mathcal{G}(\theta, T, X)$. There are an open subset U of X contained in D_{u^*u} such that $[u, x] \in \Theta\left(u, U\right) \subset W$. Pick $c \in C$ such that $x \in D_{(u\wedge c)^*(u\wedge c)}$, and an open neighborhood V of x whose closure $\overline{V}$ is contained in $U \cap D_{(u\wedge c)^*(u\wedge c)}$. We claim that $\Theta\left(u \wedge c, V\right)$ is an open neighborhood of $[u, x]$ whose closure is contained in W. To show this it is enough to show that $\Theta\left(u \wedge c, \overline{V}\right)$ is closed in $\mathcal{G}(\theta, T, X)$. Pick $[v, y] \in \mathcal{G}(\theta, T, X)\backslash\Theta\left(u \wedge c, \overline{V}\right)$. If $y \notin \overline{V}$ then clearly there is an open neighborhood of $[t, y]$ disjoint from $\Theta\left(u \wedge c, \overline{V}\right)$. Suppose that $y \in \overline{V}$. Pick $d \in C$ such that $y \in D_{(u\wedge d)^*(u\wedge d)}$. In such case we have that
$$\Theta\left(u \wedge d, D_{(u\wedge d)^*(u\wedge d)}\right)$$
is an open neighborhood of y disjoint from $\Theta\left(u \wedge c, \overline{V}\right)$. This concludes the proof. □

7.5. **Étale groupoids as groupoids of germs.** Suppose that G is an étale Polish groupoid, and Σ is a countable inverse semigroup of open bisections of G. One can define the standard action of Σ on G^0 by setting $D_e = e$ for every $e \in E(\Sigma)$, and $\theta_u : D_{u^*u} \to D_{uu^*}$ by
$$\theta_u(x) = r(ux),$$
where ux is the only element of u with source x. The same proof as [10, Proposition 5.4] shows the following fact:

Proposition 7.5.1. *Suppose that Σ is a countable inverse semigroup of open bisections of G such that $\bigcup \Sigma = G$ and for every $u, v \in \Sigma$, $u \cap v$ is the union of the elements of Σ contained in $u \cap v$. Consider the standard action $\theta : \Sigma \curvearrowright G^0$. The map from $\mathcal{G}(\theta, \Sigma, X)$ to G assigning to the germ $[u, x]$ of u at x the unique element of u with source x is well defined, and it is an isomorphism of étale Polish groupoids.*

In particular every étale Polish groupoid is isomorphic to the groupoid of germs of an action of an inverse semigroup on a Polish space.

7.6. **Borel bisections.** We will say that a (standard) Borel groupoid is countable if for every $x \in G^0$, the set $Gx = s^{-1}[\{x\}]$ is countable. Observe that the countable Borel equivalence relations are exactly the principal countable Borel groupoids.

Suppose that G is a countable Borel groupoid. Observe that the set $\mathcal{S}(G)$ of Borel bisections of G is an inverse semigroup. The idempotent semilattice $E(S)$ is the Boolean algebra of Borel subsets of G^0. The order $\leq$ on $\mathcal{S}(G)$ as in Subsection 7.4 is defined by $u \leq v$ iff $u \subset v$. Therefore (S) is a semilattice with $u \wedge v = u \cap v$.

Lemma 7.6.1. *Suppose that X, Z are standard Borel spaces and $s : Z \to X$ is a Borel countable-to-one surjection. There is a countable partition $(P_n)_{n\in\omega}$ of Z into Borel subsets such that $s_{|P_n}$ is 1:1 for every $n \in \omega$.*

Proof. It is enough to show that $Z = \bigcup_n P_n$, where P_n are Borel subsets of Z such that $s_{|P_n}$ is 1:1. After replacing Z with the disjoint union of Z and $X \times \omega$, and setting $s(x, n) = x$ for $(x, n) \in X \times \omega$, we can assume that for every $x \in X$ the inverse image $s^{-1}\{x\}$ is countably infinite. We want to define a Borel function $e : X \to Z^\omega$ such that $\{e(x)_n : n \in \omega\}$ is an enumeration of $s^{-1}\{x\}$ for every $x \in X$. Consider the Borel subset E of $X \times Z^\omega$ defined by

$$\begin{aligned} (x, (e_n)) \in E \quad &\Leftrightarrow \quad (e_n) \text{ is a enumeration of } s^{-1}\{x\} \\ &\Leftrightarrow \quad s(e_n) = x \text{ and } \forall z \in s^{-1}\{x\} \ \exists n \text{ such that } z = e_n. \end{aligned}$$

(Recall that the image of a standard Borel space under a countable-to-one Borel function is Borel; see [28, Exercise 18.15].) We want to find a Borel uniformization of E. For each $x \in X$ endow $s^{-1}\{x\}$ with the discrete topology and $s^{-1}\{x\}^\omega$ with the product topology. Observe that for $(e_n) \in s^{-1}\{x\}^\omega$ we have that $(e_n) \in E_x$ iff $\forall z \in s^{-1}\{x\}$ $\exists n \in \omega$ such that $e_n = z$. Thus E_x is a dense G_δ subset of $s^{-1}\{x\}^\omega$. Define the following σ-ideal $\mathcal{I}_x$ in Z^ω: $A \in \mathcal{I}_x$ iff $A \cap E_x$ is meager in $s^{-1}\{x\}^\omega$. Thus $E_x \notin \mathcal{I}_x$. In order to conclude that E has a Borel uniformization, by [28, Theorem 18.6] it is enough to show that the assignment $x \mapsto \mathcal{I}_x$ is Borel-on-Borel as in [28, Definition 18.5]. Suppose that Y is a standard Borel space and $A \subset Y \times X \times Z^\omega$. Consider the set

$$\begin{aligned} &\{(y, x) \in Y \times X : A_{y,x} \in \mathcal{I}_x\} \\ = \ &\left\{(y, x) \in Y \times X : A_{y,x} \cap E_x \text{ is meager in } s^{-1}\{x\}^\omega\right\} \end{aligned}$$

Clearly we can assume that $A \subset Y \times E$. If $e : \omega \to s^{-1}\{x\}$ is a bijection, then e induces a homeomorphism $\pi_e : \omega^\omega \to s^{-1}\{x\}^\omega$. Therefore for $(y, x) \in Y \times X$ we have that

$$\begin{aligned} A_{y,x} \cap E_x \text{ is meager in } s^{-1}\{x\}^\omega \quad &\Leftrightarrow \quad \pi_e^{-1}\left[A_{y,x} \cap s^{-1}\{x\}^\omega\right] \text{ is meager} \\ &\Leftrightarrow \quad \{w \in \omega^\omega : \pi_e(w) \in A_{y,x}\} \text{ is meager.} \end{aligned}$$

Consider the Borel subset Q of $Y \times X \times Z^\omega$ defined by $(y, x, e) \in Q$ iff $(x, e) \in E$ and $\forall n, m \in \omega$ if $n \neq m$ then $e_n \neq e_m$ and $\{w \in \omega^\omega : (y, x, e \circ w) \in A_{y,x}\}$ is meager. We have that

$$\begin{aligned} A_{y,x} \in \mathcal{I}_x \quad &\Leftrightarrow \quad \exists e \text{ such that } (y, x, e) \in Q \\ &\Leftrightarrow \quad \forall e \forall n \neq m \in \omega,\ (x, e) \in E, \text{ and } e_n \neq e_m \ \Rightarrow \ (z, x, e) \in Q. \end{aligned}$$

This shows that $\{(y, x) : A_{y,x} \in \mathcal{I}_x\}$ is both analytic and co-analytic, and hence Borel. ☐

Proposition 7.6.2. *If G is a countable Borel groupoid, then there is a countable partition of G into Borel bisections. Moreover for every $n \in \omega$ we have that*

$$\left\{x \in G^0 : |Gx| = n\right\}$$

is Borel.

Proof. The source map $s : G \to G^0$ satisfies the hypothesis of Lemma 7.6.1. Therefore one can find a countable partition $\mathcal{H}$ of G into Borel subsets such that the source map is 1:1 on every element of $\mathcal{H}$. Define

$$\mathcal{C} = \left\{u \cap v^{-1} : u, v \in \mathcal{H}\right\}$$

an observe that $\mathcal{C}$ is a countable collection of pairwise disjoint Borel bisections of G. Observe now that for every $u \in \mathcal{C}$,

$$\left\{x \in G^0 : \exists \gamma \in u,\, x = s(\gamma)\right\} = s[u] = u^{-1}u$$

is Borel being 1:1 image of a Borel set. Moreover $|Gx| = m$ iff $\exists u_0, \ldots, u_{m-1} \in \mathcal{C}$ pairwise distinct such that $x \in u_i u_i^{-1}$ for $i \in m$ and $\forall w \in \mathcal{C}$ if $x \in ww^{-1}$ then $w = u_i$ for some $i \in m$. ☐

Let us say that a Borel bisection u is *full* if $uu^{-1} = u^{-1}u = G^0$. It is clear from Proposition 7.6.2 that if G is a countable Borel groupoid, then there is a partition of G into full Borel bisections.

7.7. **A Polish topology on countable Borel groupoids.** In this subsection we observe that any countable Borel groupoid is Borel isomorphic to a regular zero-dimensional étale Polish groupoid. Suppose that G is a countable Borel groupoid. Pick a countable partition C of G into full Borel bisections and consider the smallest inverse subsemigroup of T with the property that $u \cap v \in T$ whenever $u, v \in T$. Observe that T is countable. By [28, Exercise 13.5] there is a zero-dimensional Polish topology τ^0 on G^0 generating the Borel structure on G^0 such that $u^{-1}u$ is clopen for every $u \in T$. Consider the standard action θ of T on $\left(G^0, \tau^0\right)$ and observe that it satisfies the condition of Proposition 7.4.1. Therefore the associated groupoid of germs $\mathcal{G}\left(\theta, T, G^0\right)$ is an étale zero-dimensional regular Polish groupoid. Arguing as in the proof of [10, Proposition 5.4] one can verify that the function ϕ from G to $\mathcal{G}\left(\theta, T, G^0\right)$ sending γ to $[c, s(\gamma)]$ where c is the only element of C such that $\gamma \in C$ is a well defined Borel isomorphism of countable Borel groupoids.

7.8. **Treeable Borel groupoids.** Suppose that G is a countable Borel groupoid. A *graphing* Q of G is a Borel subset Q of $G \backslash G^0$ such that $Q = Q^{-1}$ and $\bigcup_{n\in\omega} Q^n = G$, where $Q^0 = G^0$. Suppose that Q is a graphing of G. Define $P^*(Q)$ to be the set of finite nonempty sequences $(\gamma_i)_{i\in n+1}$ in Q such that $r(\gamma_{i+1}) = s(\gamma_i)$ and $\gamma_{i+1} \neq \gamma_i^{-1}$ for $i \in n$. For $(\gamma_i)_{i\in n+1}$ in $P^*(Q)$ one can define

$$\prod_{i\in n+1} \gamma_i$$

to be the product $\gamma_n\gamma_{1n-2}\cdots\gamma_1\gamma_0$ in G. We say that Q is a *treeing* if for every $(\gamma_i)_{i\in n+1} \in P^*(Q)$,

$$\prod_{i\in n+1} \gamma_i \notin Q^0$$

or, equivalently, for every $\gamma \in G\backslash G^0$ there is exactly one element $(\gamma_i)_{i\in n+1}$ of $P^*(Q)$ such that $\prod_{i\in n+1}\gamma_i = \gamma$. A countable Borel groupoid is *treeable* when it admits a treeing [1, Section 8].

It is not difficult to verify that a principal countable Borel groupoid is treeable precisely when it is treeable as an equivalence relation. A countable group is treeable as groupoid if and only if it is a free group.

In the following if Q is a treeing of G we denote by $P(Q)$ the union of $P^*(Q)$ and $\{\varnothing\}$. In analogy with free groups, if $(\gamma_n, \ldots, \gamma_0) \in P(Q)$ we say that $\gamma_n\cdots\gamma_0$ is a *reduced word*, and that the length $l(\gamma_n\cdots\gamma_0)$ of $\gamma_n\cdots\gamma_0$ is $n+1$.

Proposition 7.8.1. *Suppose that G is a countable Borel groupoid. If there is a Borel complete section A for E_G such that $G_{|A}$ is treeable, then G is treeable.*

Proof. Pick a Borel function $f : G^0 \to G$ such that $f(a) = a$ for $a \in A$, $s\left(f(x)\right) = x$ and $r\left(f(x)\right) \in A$ for $x \in G^0$. Suppose that Q_A is a treeing for $G_{|A}$. Observe that $Q_A \cup f\left[G^0 \backslash A\right]$ is a treeing for G. □

We want to show that Borel subgroupoids of treeable groupoid are treeable. A particular case of this statement is that a subgroup of a countable free group is free, which is the well known *Nielsen-Schreier theorem.* The strategy of our proof will be a Borel version for groupoids of Schreier's proof of the Nielsen-Schreier theorem.

Suppose that G is a treeable groupoid with no elements of order 2, and H is a Borel subgroupoid of G. In the rest of the subsection we will show that H is treeable. Denote by $\sim_H$ the equivalence relation $\gamma \sim_H \rho$ iff $\gamma H = \rho H$.

Suppose that Q is a treeing for G. Since G has no elements of order 2 we can write $Q=Q^{+}\cup Q^{-}$ where Q^{+} and Q^{-} are disjoint and $Q^{+}=\left(Q^{-}\right)^{-1}$. A Borel transversal U for $\sim_H$ is *Schreier* if $\gamma_n\cdots\gamma_0\in T$ implies $\gamma_k\cdots\gamma_0\in T$ for $k\in n$. We want to show that there is a Schreier Borel transversal for H.

Suppose that $(V_n)_{n\in\omega}$ is a partition of $G\backslash G^0$ into full Borel bisections. If $\gamma_n\cdots\gamma_0$ and $\gamma'_m\cdots\gamma'_0$ are reduced words with $r(\gamma_n)=r\left(\gamma'_m\right)=x$, set

$$\gamma_n\cdots\gamma_0<_x\gamma'_m\cdots\gamma'_0$$

iff $n<m$, or $n=m$ and for some $k\in n$, $\gamma_i=\gamma'_i$ for $i\in k$ and for some $N\in\omega$, $\gamma_k\in V_N$ while $\gamma'_k\notin V_n$ for any $n\leq N$. Define also

$$x<_x\gamma_n\cdots\gamma_0.$$

Observe that $<_x$ is a Borel order of xG with minimum x, and the function $x\mapsto<_x$ is Borel. Define now for $\gamma\in G$, $\overline{\gamma}$ to be the $<_{r(\gamma)}$-least element of γH. Thus $\overline{\gamma}\in\gamma H$ and hence $\overline{\gamma}^{-1}\gamma\in H$. Consider $U=\left\{\overline{\gamma}^{-1}\gamma:\gamma\in G\right\}$ and observe that, since x is the $<_x$-minimum element of xG, $U\cap H\subset H^0$. Arguing as in [25, Section 2.3] one can show that U is a Schreier transversal for $\sim_H$. Define then

$$A=\left\{\overline{\gamma u}^{-1}\gamma u:u\in U,\,\gamma\in Q\right\}\subset H.$$

The same proof as Lemma 3 in [25, Section 3.3] shows that $\bigcup_{n\in\omega}A^n=H$. Define now

$$B=\left\{\overline{\gamma u}^{-1}\gamma u:u\in U,\,\gamma\in Q^{+},\text{ and }\gamma u\notin U\right\}.$$

The same proof as Lemma 4 in [25, Section 3.4] shows that

$$B^{-1}=\left\{\overline{\gamma u}^{-1}\gamma u:u\in U,\,\gamma\in Q^{-},\text{ and }\gamma u\notin U\right\},$$

and $A\backslash H^0$ is the disjoint union of B and B^{-1}. Finally one can show that $A\backslash H^0$ is a treeing for G as in [25, Section 3.6]. The proof is the same as the proof of Theorem 1 in [25, Section 3.6]. The fundamental lemma is the following:

Lemma 7.8.2. *Suppose that $b=\overline{u\gamma}^{-1}u\gamma\in A\backslash H^0$ and $b'=\overline{v\rho}^{-1}v\rho\in A\backslash H^0$. The product $\rho v\overline{\gamma u}^{-1}\gamma$ is equal to a reduced word $\rho w\gamma$ for some $w\in G$, unless $v=\overline{\gamma u}$ and $\rho=\gamma^{-1}$, in which case*

$$u=\overline{\gamma^{-1}\overline{\gamma u}}=\overline{\rho v}$$

and

$$b'=b^{-1}.$$

The proof of Lemma 7.8.2 is analogous to the proof of Lemma 5 in [25, Section 3.5].

8. Functorial Borel complexity and treeable equivalence relations

8.1. The lifting property.

Definition 8.1.1. Suppose that G is a Polish groupoid. We say that G has the *lifting property* if the following holds: For any Polish groupoid H such that E_H is Borel, and any Borel function $f : G^0 \to H^0$ such that $f(x)E_H f(x')$ whenever $xE_G x'$, there is a Borel functor $F : G \to H$ that extends f.

Remark 8.1.2. If E_G has the lifting property (as a principal groupoid), then G has the lifting property.

Proposition 8.1.3. *A treeable countable Borel groupoid with no elements of order* 2 *has the lifting property.*

Proof. Suppose that G is a treeable countable Borel groupoid with no elements of order 2, H is a Polish groupoid such that E_H is Borel, and $f : G^0 \to H^0$ is a Borel function such that $f(x)E_G f(x')$ whenever $xE_G x'$. Suppose that Q is a treeing for G. Write $Q = Q^+ \cup Q^-$ where $Q^+ = (Q^-)^{-1}$ and Q^+ and Q^- are disjoint. Since E_H is Borel, then map $(x, y) \mapsto xHy$ from E_G to $F(H)\setminus\{\varnothing\}$ is Borel by Theorem 5.2.2. Fix a Borel map $\sigma : F(H)\setminus\{\varnothing\} \to H$ such that $\sigma(A) \in A$ for every $A \in F(H)\setminus\{\varnothing\}$. Define

- $F(x) = f(x)$ for $x \in G^0$,
- $F(\gamma) = \sigma\left(f(r(\gamma))Hf(s(\gamma))\right)$ for $\gamma \in Q^+$,
- $F(\gamma) = F\left(\gamma^{-1}\right)^{-1}$ for $\gamma \in (Q^+)^{-1}$, and
- $F(\gamma_n \cdots \gamma_0) = F(\gamma_n) \cdots F(\gamma_0)$ if $\gamma_n \cdots \gamma_0 \in G\backslash G^0$ is a reduced word.

It is immediate to check that F is a Borel functor such that $F_{|G^0} = f$. $\square$

Proposition 8.1.4. *If G is a Polish groupoid and $A \subset G^0$ is a Borel complete section for E_G such that $G_{|A}$ has the lifting property and there is a Borel map $\phi : G^0 \to G$ such that $s(\phi(x)) = x$ and $r(\phi(x)) \in A$ for every $x \in G^0$, then G has the lifting property.*

Proof. Without loss of generality we can assume that $\phi(x) = x$ for $x \in A$. Define $y(x) = r(\phi(x))$ for $x \in G^0$. Suppose that $f : G^0 \to H^0$ is a Borel function such that $f(x)E_H f(x')$ whenever $xE_G x$. Since $G_{|A}$ ha the lifting property there is a Borel functor $F : G_{|A} \to H$ such that $F_{|A} = f_{|A}$. Define $h(x) = \sigma\left(f\left(y(x)\right)Hf(x)\right)$. Define now for $\rho \in G$ such that $s(\rho) = x$ and $r(\rho) = y$

$$F(\rho) = h(y)^{-1}F\left(\phi(y)\rho\phi(x)^{-1}\right)h(x)$$

and observe that F is a Borel functor such that $F_{|G^0} = f$. $\square$

Theorem 8.1.5. *Suppose that G is a Polish groupoid. If E_G is essentially treeable, then E_G has the lifting property.*

Proof. Observe that the assignment $[x]_{E_G} \mapsto I_{[x]_{E_G}}$,where

$$A \in I_{[x]_{E_G}} \Leftrightarrow \{\gamma \in xG : s(\gamma) \in A\} \text{ is meager}$$

is a Borel ccc assignment of σ-ideals in the sense of [27, page 285]; see Subsection 2.10. It follows from [27, Theorem 1.5] together with the fact that E_G is essentially treeable that there is a countable Borel subset A of G^0 meeting every orbit in a countable nonempty set. Thus $(E_G)_{|A}$ is treeable equivalence relation. In particular by Proposition 8.1.3 the equivalence relation $(E_G)_{|A}$ has the lifting property. Therefore $G_{|A}$ has the lifting property. Since $(E_G)_{|A}$ is countable one can find a Borel map $p : X \to A$ such that $(x, p(x)) \in E_G$ for every $x \in X$ and $p(x) = x$ for $x \in A$. It follows from Proposition 8.1.4 that E_G has the lifting property. □

Corollary 8.1.6. *Suppose that G and H are Polish groupoids. If E_G is essentially treeable, and E_H is Borel, then $G \leq_B H$ if and only if $E_G \leq E_H$.*

Proposition 8.1.7. *Suppose that G is a Polish groupoid. If E_G is essentially countable, then there is an invariant dense G_δ set $C \subset G^0$ such that $(E_G)_{|C}$ is essentially hyperfinite.*

Proof. By [22, Theorem 6.2] there is a comeager and invariant subset C_0 of G^0 such that $(E_G)_{|C_0}$ is essentially hyperfinite. Pick a dense G_δ subset C_1 of C_0 and then define

$$C = \{x \in X : \forall^* \gamma \in Gx,\, \gamma x \in C_1\}.$$

The properties of the Vaught transform together with Lemma 2.9.1 imply that C is an invariant dense G_δ set contained in C_0. In particular $(E_G)_{|C}$ is essentially hyperfinite. □

Corollary 8.1.8. *Suppose that G is a Polish groupoid such that E_G is essentially countable. There is an invariant dense G_δ subset C of G^0 with the following property: For any Polish groupoid H such that $E_G \leq_B E_H$ and E_H is Borel, $G_{|C} \leq_B H$.*

8.2. **The cocycle property.**

Definition 8.2.1. An analytic groupoid G has the *cocycle property* if there is a Borel functor $F : E_G \to G$ such that $F(x, x) = x$ for every $x \in G^0$.

It is immediate to verify that a Polish group action $G \curvearrowright X$ has the cocycle property as defined in [22] if and only if the action groupoid $G \ltimes X$ has the cocycle property as in Definition 8.2.1. The proof of the following proposition is essentially the same of the proof of the implication (ii)⇒(iii) in [24, Theorem 3.7], and it is presented for convenience of the reader.

Proposition 8.2.2. *Suppose that G is a countable Borel groupoid, and X a Borel G-space. If $G \ltimes X$ has the cocycle property, then there is a free Borel G-space Y such that $E_G^Y \sim_B E_G^X$. Moreover if G is treeable then E_G^X is treeable.*

Proof. Since $G \ltimes X$ has the cocycle property there is a Borel functor

$$F : E_G^X \to G$$

such that $s(F(x,y)) = p(y)$ and $F(x,y)y = x$. Consider the equivalence relation $\sim$ on $G \ltimes X$ defined by $(\gamma, x) \sim (\rho, y)$ iff $(x,y) \in E_G^X$ and $\gamma F(x,y) = \rho$. Clearly $\sim$ is Borel. We now show that it has a Borel selector. Observe that the range H of F is a Borel subgroupoid of G (since F is countable to one). By Proposition 5.1.1 there is a Borel selector $t : G \to G$ for the equivalence relation $\gamma \sim_H \gamma'$ iff $\gamma H = \gamma' H$. Observe that if $(\gamma, x) \sim (\rho, y)$ then $\gamma H = \rho H$ and hence $t(\gamma) = t(\rho)$. Moreover there is a unique element x_0 of X such that $(t(\gamma), x_0) \sim (\gamma, x)$. Define $S(\gamma, x) = (t(\gamma), x_0)$ and observe that S is a Borel selector for the equivalence relation $\sim$. Define Y to be the quotient of $G \ltimes X$ by $\sim$. Define now the Borel action of G on Y by $p[\gamma, x] = r(\gamma)$ and $\rho[\gamma, x] = [\rho\gamma, x]$ for $\rho \in Gr(\gamma)$. It is easy to verify that such an action is free, and $[\gamma, x] \, E_G^Y \, [\rho, y]$ iff $x E_G^X y$. Let us now observe that $E_G^X \sim_B E_G^Y$. If $q : X \to G$ is a Borel map such that $s(q(x)) = p(x)$ for every $x \in X$, then the map $x \mapsto [q(x), x]$ is a Borel reduction from E_G^X to E_G^Y. Conversely the map $[\gamma, x] \mapsto x^*$ where $[t(\gamma), x^*] = S(\gamma, x)$ is a Borel reduction from E_G^Y to E_G^X. Suppose finally that G is treeable with treeing Q. We want to show that E_G^X is treeable. Since $E_G^X \sim_B E_G^Y$, it is enough to show that E_G^Y is treeable. Fix an equivalence class $[[\gamma, x]]_F$ of E_G^Y. Observe that the map from $[[\gamma, x]]_{E_G^Y}$ to $Gp(x)$ defined by $[\rho, y] \mapsto \rho F(y, x)$ is bijective. One can then consider the treeing

$$\{[\rho, y] \in Y : \rho F(y, x) \in Q\}$$

for E_G^Y. □

Lemma 8.2.3 can be proved similarly as Proposition 8.1.4.

Lemma 8.2.3. *Suppose that G is a countable Borel groupoid action, and $A \subset G^0$ is a Borel complete section for E_G. If there is a Borel function $\psi : (E_G)_{|A} \to G$ such that $s(\psi(x,y)) = y$ and $r(\psi(x,y)) = x$, then G has the cocycle property.*

Lemma 8.2.4. *Suppose that G is a countable Borel groupoid. If $E_G \leq_B G$, then there is an invariant Borel subset Y of G^0 such that $G_{|Y}$ has the cocycle property and $(E_G)_{|Y} \sim_B E_G$*

Proof. Since $E_G \leq_B G$ there is a Borel functor $F : E_G \to G$. Define $A \subset G^0$ to be image of X under F. Since $F_{|X}$ is countable-to-one, A is a Borel subset of G^0; see [28, Theorem 18.10]. By [28, Exercise 18.14] there is a Borel function $g : A \to X$ such that $(f \circ g)(y) = y$ for every $y \in A$. Define now Y to be the union of the orbits of G that meet A. Clearly $E_G \sim_B E_{G_{|Y}}$. The map

$$\begin{array}{rcl} (E_G)_{|A} & \to & G \\ (x,y) & \mapsto & F(g(x), g(y)) \end{array}$$

together with Lemma 8.2.3 imply that $G_{|Y}$ has the cocycle property. □

8.3. **Free actions of treeable groupoids.** We want to show that, if G is a treeable groupoid, and $G \curvearrowright X$ is a free Borel action of G, then the associated orbit equivalence relation is treeable. This will follow from a more general result about $\mathcal{L}$-structured equivalence relations.

Suppose that $\mathcal{L} = \{R_n : n \in \omega\}$ is a countable relational language in first order logic, where R_n has arity $k_n \in \omega$. Suppose that E is a countable Borel equivalence relation on a standard Borel space X. According to [24, Definition 2.17] the equivalence relation E is $\mathcal{L}$-structured if there are Borel relations $R_n^E \subset X^{k_n}$ such that, for any $k \in \omega$ and $x_1, \ldots, x_{k_n} \in X$, $x_1, \ldots, x_{k_n}$ belong to the same E-class whenever $(x_1, \ldots, x_{k_n}) \in R_n^E$. In particular every E-class $[x]$ is the universe of an $\mathcal{L}$-structure

$$\left\langle [x], \left([x]^{k_n} \cap R_n^E \right)_{n \in \omega} \right\rangle .$$

Similarly, if X is a standard Borel space, then *standard Borel bundle $\mathcal{A}$ of countable $\mathcal{L}$-structures* over X is a standard Borel space $\mathcal{A}$ fibred over X with countable fibers $(A_x)_{x \in X}$, endowed with Borel subsets $R_n^{\mathcal{A}} \subset \mathcal{A}^{k_n}$ such that, for any $k \in \omega$ and $a_1, \ldots, a_{k_n} \in \mathcal{A}$, $a_1, \ldots, a_{k_n}$ belong to the same fiber over X whenever $(a_1, \ldots, a_{k_n}) \in R_n^{\mathcal{A}}$. Suppose that G is a Polish groupoid, and $\mathcal{A}$ is a standard Borel of $\mathcal{L}$-structures over G^0. A Borel action $G \curvearrowright \mathcal{A}$ is a Borel map $(\gamma, a) \mapsto \gamma a$ defined for $(\gamma, a) \in \mathcal{A} \times G$ such that $a \in A_{s(\gamma)}$, and $(a_1, \ldots, a_{k_n}) \in R_n^{\mathcal{A}}$ if and only if $(\gamma a_1, \ldots, \gamma a_{k_n}) \in R_n^{\mathcal{A}}$ for every $n \in \omega$, $\gamma \in G$, and $a_1, \ldots, a_{k_n} \in A_{s(\gamma)}$. The proof of the following theorem is very similar to the argument at the beginning of Section 3.2 in [24], and it is reproduced for convenience of the reader.

Theorem 8.3.1. *Suppose that G is a Polish groupoid such that there is a standard Borel bundle $\mathcal{A}$ of countable $\mathcal{L}$-structures and a Borel action $G \curvearrowright \mathcal{A}$ such that the corresponding orbit equivalence relation $E_G^{\mathcal{A}}$ is smooth, and for every $a \in \mathcal{A}$ the stabilizer $G_a = \{\gamma \in Gp_{\mathcal{A}}(a) : \gamma a = a\}$ is a compact subset of G. If X is a standard Borel space, and $G \curvearrowright X$ is a free Borel action of G on X, then there is an $\mathcal{L}$-structured countable Borel equivalence relation E such that $E \sim_B E_G^X$, and moreover every class of E is isomorphic to some fiber of $\mathcal{A}$.*

Proof. By Corollary 2.10.9 there is a Borel selector t for $E_G^{\mathcal{A}}$. Moreover by Theorem 4.1.1 we can assume without loss of generality that the action $G \curvearrowright X$ is continuous. Define

$$\mathcal{A} * X = \left\{ (x, a) \in X \times A : a \in A_{p(x)} \right\}$$

and the action $G \curvearrowright (X * A)$ by $\gamma (x, a) = (\gamma x, \gamma a)$. Observe that such an action is free and in particular the associated orbit equivalence relation $\sim$ is Borel. We now show that $\sim$ has a Borel selector. Suppose that $(x, a) \in X * A$. Observe that if $(x, a) \sim (x, b)$ then $a E_G^{\mathcal{A}} b$ and hence $t(a) = t(b)$. Therefore

$t(a)$ depends only on the $\sim$-class $[x,a]$ of (x,a). Observe that

$$G_{t(a)}x = \{y \in X : (y, t(a)) \in [x,a]\}$$

is a compact subset of X. Denote by $\sigma : F(X)\backslash\{\varnothing\} \to X$ a Borel function such that $\sigma(A) \in A$ for every $A \in F(X)\backslash\{\varnothing\}$. Define $S(x,a) = \left(\sigma\left(G_{t(a)}x\right), t(a)\right)$, and observe that S is a Borel selector for $\sim$. Define the standard Borel space $Y = (X * A) / \sim$ and the countable Borel equivalence relation E on Y by $[x,a]\, E\, [y,b]$ iff $xE_G^X y$. We now define for every E-class $C = [[x,a]]_E$ an $\mathcal{L}$-structure on C. Fix $n \in \omega$ and suppose that $[\gamma_i x, a_i]$ for $i \in k_n$ are elements of C, where $\gamma_i \in Gp(x)$ for $i \in k_n$. Set $([\gamma_i x, a_i])_{i \in k_n} \in R_n^C$ iff $\left(\gamma_i^{-1} a_i\right)_{i\in k_n} \in R_n^{\mathcal{A}}$. Using the fact that G acts by $\mathcal{L}$-isomorphisms one can verify that this does not depend on the choice of $[x,a] \in C$. Define now $([x_i,a_i])_{i\in k_n} \in R_n^E$ if and only if $[x_0,a_0]\, E\, [x_1,a_1]\, E \cdots E\, [x_{k_n-1}, a_{k_n-1}]$ and $([x_i,a_i])_{i\in k_n} \in R_n^C$ where $C = [[x,a]]_E$. This defines Borel relations R_n^E on E that make E $\mathcal{L}$-structured. Moreover the Borel map $f : C \to \mathcal{A}$ defined by $f[\gamma x, a] = \gamma^{-1} a$, where C is the class of $[x,a]$, shows that the $\mathcal{L}$-structure

$$\left\langle C, \left(R_n \cap C^{k_n}\right)_{n\in\omega}\right\rangle$$

is isomorphic to $A_{p(x)}$. Finally we observe now that $E \sim E_G^X$. If $q : G^0 \to \mathcal{A}$ such that $q(x)$ belongs to the fiber A_x for every $x \in G^0$, then the map $x \mapsto [x, q(p(x))]$ is a Borel reduction from E_G^X to E. Conversely the map $[x,a] \mapsto x^*$ where $[x^*, a^*] = S([x,a])$ witnesses that E is Borel reducible to E_G^X. □

Let us now consider the particular case of Theorem 8.3.1 when $\mathcal{L}$ is the language with a single binary relation. Assume further that G is a treeable Borel groupoid. A standard Borel bundle of trees over X is a standard Borel bundle $(A_x)_{x\in X}$ of countable $\mathcal{L}$-structures such that A_x is a tree for every $x \in X$. A treeing of G defines on G a structure of standard Borel bundle of trees over G. Moreover the action of G on itself by left translation is compatible with such a bundle of trees structure, and has a smooth orbit equivalence relation. Therefore by Theorem 8.3.1 E is treeable. This concludes the proof of the following corollary.

Corollary 8.3.2. *If G is a countable treeable groupoid, and $G \curvearrowright X$ is a free Borel action, then the orbit equivalence relation E_G^X is treeable.*

8.4. **Characterizing treeable equivalence relations.** Denote by $\mathbb{F}_\infty$ the free countable group on infinitely many generators. The following result subsumes [24, Theorem 3.7].

Theorem 8.4.1. *Suppose that E is a countable Borel equivalence relation on a standard Borel space X. The following statements are equivalent:*

1. *E is treeable;*
2. *E has the lifting property;*

(3) *For every countable Borel groupoid G and Borel action $G \curvearrowright X$ such that $E_G^X = E$, the groupoid $G \ltimes X$ has the cocycle property;*
(4) *For every Borel action $\mathbb{F}_\infty \curvearrowright X$ such that $E_{\mathbb{F}_\infty}^X = E$, $E_{\mathbb{F}_\infty}^X \leq_B \mathbb{F}_\infty \ltimes X$;*
(5) *For every countable Borel groupoid G and Borel action $G \curvearrowright X$ such that $E_G^X = E$, there is a free Borel action $G \curvearrowright Y$ such that $E_G^Y \sim E_G^X$;*
(6) *For every countable Borel groupoid G and Borel action $G \curvearrowright X$ such that $E \subset E_G^X$ there is a free Borel action $G \curvearrowright Y$ such that $E \sqsubseteq_B E_G^Y$.*

Proof. **(1)⇒(2):** It follows from Proposition 8.1.3.

(2)⇒(3): It follows form the fact that if E_G^X has the lifting property, then $G \ltimes X$ has the cocycle property.

(3)⇒(4): Obvious.

(4)⇒(1): It follows from Lemma 8.2.4 and Proposition 8.2.2.

(3)⇒(5): This follows from Proposition 8.2.2.

(5)⇒(1): Consider an action $\mathbb{F}_\infty \curvearrowright X$ such that $E = E_{\mathbb{F}_\infty}^X$ and then apply (5) and Corollary 8.3.2.

(2)⇒(6): Since E has the lifting property, there is a Borel function $F : E \to G$ such that $s(F(x, y)) = p(y)$ and $F(x, y)y = x$ for every $(x, y) \in E$. Consider on $G \ltimes X$ the equivalence relation $(\gamma, x) \sim (\rho, y)$ iff xEy and $\rho = \gamma F(x, y)$. Proceeding as in the proof of Proposition 8.2.2 one can show that $\sim$ has a Borel selector. Thus the quotient Y of $G \ltimes X$ by $\sim$ is standard. Define the Borel action $G \curvearrowright Y$ by $p[\gamma, x] = r(\gamma)$ and $\rho[\gamma, x] = [\rho\gamma, x]$. As in the proof of Proposition 8.2.2 one can show that such an action is free. Moreover the map $x \mapsto [p(x), x]$ is an injective Borel reduction from E to E_G^Y.

(6)⇒(1): It follows from Corollary 8.3.2 together with the fact that a subrelation of a treeable equivalence relation is treeable [24, Proposition 3.3]; see also Subsection 7.8.

□

The following corollary is an immediate consequence of the implication (4)⇒(1) in Theorem 8.4.1.

Corollary 8.4.2. *For any nontreeable equivalence relation E there are countable Borel groupoids G and H that have E as orbit equivalence relation such that G is not Borel reducible to H. Moreover one can take $G = E$ and H to be the action groupoid of an action of $\mathbb{F}_\infty$.*

Corollary 8.4.2 can be interpreted as asserting that functorial Borel complexity provides a finer distinction between the complexity of classification problems in mathematics than the traditional notion of Borel complexity for equivalence relations

9. Further directions and open problems

Polish groupoids seem to be a broad generalization of Polish groups and Polish group actions. For instance, as shown in Section 4, any Borel action of a Polish groupoid is again a Polish groupoid. However, no example is currently known of a Polish groupoid whose orbit equivalence relation is not Borel bireducible with an orbit equivalence relation of a Polish group action.

Problem 9.1. Is there a Polish groupoid G such that the orbit equivalence relation E_G is not Borel bireducible with an orbit equivalence relation of a Polish group action? Can one find such a G for which E_G is moreover Borel?

Problem 9.1 has a tight connection with the a conjecture due to Hjorth and Kechris [23, Conjecture 1]. Recall that E_1 denotes the relation of tail equivalence for sequences in $[0, 1]$. Theorem 4.2 of [26] asserts that E_1 is not Borel reducible to the orbit equivalence relation of a Polish group action. The Hjorth-Kechris conjecture asserts that is E is a Borel equivalence relation then the converse holds, namely if E_1 is not Borel reducible to E then E is Borel bireducible with the orbit equivalence relation of a Polish group action. By Corollary 2.10.8 E_1 is not reducible to any Borel orbit equivalence relation of a Polish groupoid. Therefore a groupoid G as in Problem 9.1 such that E_G is moreover Borel would provide a counterexample to the Hjorth-Kechris conjecture.

In the theory of Borel complexity of equivalence relations, a key role is played by equivalence relations that are complete (or universal) for a given class up to Borel reducibility. These equivalence relations provide natural benchmarks of complexity. It would be interesting to analogously find universal elements for natural classes of analytic groupoids.

Problem 9.2. Establish whether the following classes have a universal element up to Borel reducibility: analytic groupoids, Borel groupoids, Polish groupoids, countable groupoids.

In the world of equivalence relations, the phenomenon of universality is widespread. Most natural classes of analytic equivalence relations have universal elements. Moreover such universal elements admit many different descriptions. For example the following equivalence relations are all complete for orbit equivalence relations induced by Polish group actions [9, 14, 19, 38, 45, 49]:

(1) Isomorphism of abelian C*-algebras;
(2) Isomorphism of amenable, simple, unital, separable C*-algebras;
(3) Isometry of separable Banach spaces;
(4) Complete order isomorphism of separable operator systems;
(5) Isometry of metric spaces;
(6) Conjugacy of isometries of the Urysohn sphere.

As shown in Subsections 2.6 and 2.7 the relations 1-6 above are naturally the orbit equivalence relation of a Borel groupoid. It would be interesting to known if the functorial Borel complexity of such groupoids are different.

Problem 9.3. Consider the standard Borel groupoids associated with the relations 1-6 above. Are these groupoids Borel bi-reducible?

One can consider a similar problem for the classes of countable first-order structures that are Borel complete [17]. Recall that a class of countable first-order structures is Borel complete if the corresponding isomorphism relation is complete for isomorphism relations of countable structures. Borel complete classes include countable trees, countable linear orders, countable fields of a fixed characteristic, and countable groups.

Problem 9.4. Are there two Borel complete classes of first order structures such that the corresponding groupoids are not Borel bi-reducible?

Appendix by Anush Tserunyan[1]

In this appendix we show that, if X is a locally Polish space, then the Effros Borel structure on the space $F(X)$ of closed subspaces of X is standard. Recall that, as in Definition 2.2.1, a topological space X is *locally Polish* if it has a countable basis of open sets which are Polish in the relative topology. If U is an open subset of X, we denote by U^- the set

$$\{F \in F(X) : F \cap U \neq \varnothing\}.$$

Define the Effros Borel structure on $F(X)$ to be the Borel structure generated by the sets U^- for $U \subset X$ open.

Theorem A. The Effros Borel structure on $F(X)$ is standard.

Proof. Suppose that

$$\mathcal{A} = \{U_n : n \in \omega\}$$

is a countable basis of Polish open subsets of X. For every $n \in \omega$ denote by d_n a compatible complete metric on U_n. Clearly the Effros Borel structure on $F(X)$ is generated by the sets U^- for $U \in \mathcal{A}$. Consider the collection $\mathcal{S}_{\mathcal{A}}$

$$\left\{U^-, X \backslash U^- : U \in \mathcal{A}\right\},$$

and the topology $\tau_{\mathcal{A}}$ on $F(X)$ having $\mathcal{S}_{\mathcal{A}}$ as subbasis. We will show that the topology $\tau_{\mathcal{A}}$ on $F(X)$ is Polish. Consider the map c from $F(X)$ to 2^ω assigning to F the characteristic function of $\{n \in \omega : F \cap U_n \neq \varnothing\}$. Clearly c is a $\tau_{\mathcal{A}}$-homeomorphism onto its image. In view of [28, Theorem 3.11], in order to conclude that $(F(X), \tau_{\mathcal{A}})$ is Polish it is enough to show that the image Y of c is a G_δ subspace of 2^ω. We claim that, for $y \in 2^\omega$, $y \in Y$ if and only if the following conditions hold:

(1) for all n, m with $U_n \subseteq U_m$, if $y(n) = 1$ then $y(m) = 1$;

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(2) for all n and $\varepsilon \in \mathbb{Q}_+$, if $y(n) = 1$ then there is m such that $y(m) = 1$ and for all $i \leq n$ with $U_i \supseteq U_n$, we have:

$$\overline{U}^i_m \subseteq U_n \text{ and } \mathrm{diam}_i(U_m) < \varepsilon,$$

where the closure $\overline{U}^i_m$ and diameter $\mathrm{diam}_i(U_m)$ are taken with respect to the metric d_i.

Since necessity is obvious, we check that these conditions are sufficient. Let $y \in 2^\omega$ satisfy conditions (i) and (ii), and define the $\tau_{\mathcal{A}}$-closed subset of X

$$F = \{x \in X : \forall n \in \omega,\, x \in U_n \Rightarrow y(n) = 1\}.$$

We show that $c(F) = y$. Fix $n \in \omega$ and note that if $y(n) = 0$, then $F \cap U_n = \varnothing$ by definition. So suppose $y(n) = 1$ and we have to find an $x \in F \cap U_n$. Iterating (ii), we get a sequence $(U_{n_i})_{i\in\omega}$ with $n_0 = n$ and such that for all $i \in \omega$,

- $y(n_i) = 1$,
- $\overline{U}^n_{n_{i+1}} \subseteq U_{n_i}$,
- $\mathrm{diam}_n(U_{n_i}) \leq 2^{-k}$.

Thus, since the metric d_n on U_n is complete, we get $\{x\} = \bigcap_i \overline{U}^n_{n_i}$, for some $x \in U_n$. It remains to show that $x \in F$, but this easily follows from (i). □

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